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On Arresting the Complex Growth Rates in Rotatory Triply Diffusive Convection

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Abstract

Linear stability of a triply diffusive fluid layer (one of the components may be heat) has been mathematically analyzed in the presence of uniform vertical rotation. Upper bounds for the complex growth rate of an arbitrary oscillatory perturbation of growing amplitude are derived which are important especially when at least one of the boundaries is rigid so that exact solutions in closed form are not obtainable. Further, it is proved that the results obtained herein are uniformly valid for any combination of dynamically free and rigid boundaries. It is also shown that the existing results of rotatory hydrodynamic Rayleigh Benard convection and rotatory hydrodynamic double diffusive convection follow as a consequence.

Keywords: Triply Diffusive convection; Oscillatory motions; complex growth rate; rotation

MSC 2010 No.: Primary 76E06, 76E07. Secondary 76R10, 76R50

1. Introduction

If the gradients of two stratifying agencies, such as heat and salt, with different diffusivities are simultaneously present in a fluid layer, the convective phenomena, which occurs is

known as double diffusive convection. This phenomenon is now well known and has been extensively studied (Stern (1960), Veronis (1965), Nield (1967), Baines and Gill (1969), Turner (1968, 1974), Brandt and Fernando (1996), Reena and Rana (2009) and Tyagi and Agrawal (2013) etc.). All these researchers have considered the case of two component systems. However, it has been recognized previously (Griffiths (1979), Turner (1985)) that there are many situations wherein more than two components are present. Examples of such multiple diffusive convection fluid systems include the solidification of molten alloys, Earth core, geothermally heated lakes, and magmas and their laboratory models and seawater etc. Griffiths (1979), Pearlstein et al. (1989), Moroz (1989), Lopez (1990) and Tracey (1996) have theoretically studied the onset of convection in a triply diffusive fluid layer (in porous and non-porous media, where density depends on three independently diffusing agencies with different diffusivities). The essence of the studies of these researchers is that small concentrations of a third component with a smaller diffusivity can have a significant effect upon the nature of diffusive instabilities and 'oscillatory' and direct 'salt finger' modes are simultaneously unstable under a wide range of conditions when the density gradients due to components with the greatest and smallest diffusivity are of the same signs.

Some fundamental differences between the double and triply diffusive convection are noticed by these researchers. Among these differences the first is that if the gradients of two of the stratifying agencies are held fixed, then three critical values of the Rayleigh number of the third agency are sometimes required to specify the linear stability criteria (in double diffusive convection only one critical Rayleigh number is required). The other difference is that the onset of convection for the case of free boundaries may occur via a quasi-periodic bifurcation from the motionless basic state. Terrones (1993) investigated the cross diffusion on the onset of convective instability in a horizontally unbounded, triply diffusive and triply stratified fluid layer and obtained the numerical results based on diffusivity data for the system water / potassium chloride / potassium phosphate / phosphoric acid solutions. Ryzhkov and Shevtsova (2007) studied the case of multicomponent mixture with application to the thermo-gravitational column. Ryzhkov and Shevtsova (2009) also studied the longwave instability of a multicomponent fluid layer with the soret effect. Rionero (2013a) studied a triple convective diffusive fluid mixture saturating a porous horizontal layer, heated from below and salted from above and obtained sufficient conditions for inhibiting the onset of convection and guaranteeing the global nonlinear stability of the thermal conduction solution. Rionero (2013b) also investigated the multicomponent diffusive convection in porous layer for the more general case when heated from below and salted by salts partly from above and partly from below.

The problem of obtaining bounds for the complex growth rate of an arbitrary oscillatory perturbation of growing amplitude in various hydrodynamical stability problems is an important feature of fluid dynamics, especially when both the boundaries are not dynamically free so that exact solutions in closed form are not obtainable. Banerjee et al. (1981) formulated a novel way of combining the governing equations and boundary conditions for double diffusive convection problem so that a semicircle theorem is derivable and which in turn yields the desired bounds. Since the inability of finding the exact solutions in closed form also exists for the case of three component systems when both the boundaries are not dynamically free, the bounds for the complex growth rate of an arbitrary oscillatory perturbation of growing amplitude in triply diffusive case must also be found.

The extension of Banerjee et al. (1981) result to triply diffusive convection in the domains of astrophysics and terrestrial physics, wherein the liquid concerned has the property of electrical conduction and the magnetic field and rotation are prevalent, is very much sought

after in the present context. This paper, which mathematically establishes the upper bounds for the complex growth rate of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude, in a triply diffusive fluid layer wherein a uniform rotation parallel to gravity is superimposed, may be regarded as a first step in this scheme of extended investigations.

2. Mathematical formulation and analysis

A viscous finitely heat conducting Boussinesq fluid layer of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ (rotating with uniform angular velocity $\vec{\Omega}$) which are respectively maintained at uniform temperatures T_0 and $T_1 (< T_0)$ and uniform concentrations S_{10}, S_{20} and $S_{11} (< S_{10}), S_{21} (< S_{20})$ (as shown in Figure 1). It is assumed that the cross-diffusion effects of the stratifying agencies can be neglected.

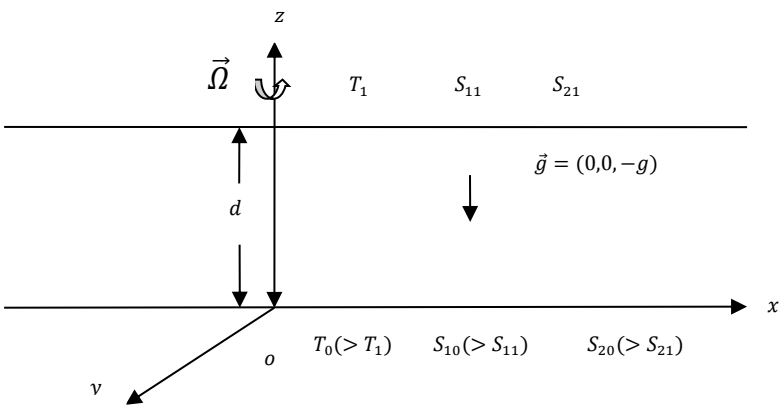

Figure 1. Physical configuration

The basic equations that govern the motions of triply diffusive fluid layer under the action of uniform vertical rotation are as follows (Prakash et al. (2014)):
Equation of continuity

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{1}$$

Equations of motion

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{P_1}{\rho_0} - \frac{1}{2}\left|\vec{\Omega} \times \vec{r}\right|^2\right) + \left(1 + \frac{\delta\rho}{\rho_0} + \frac{\delta\rho'}{\rho_0} + \frac{\delta\rho''}{\rho_0}\right) X_i$$
$$+ 2\varepsilon_{ijk} u_j \Omega_k + \nu \nabla^2 u_i. \tag{2}$$

Equation of heat conduction

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T. \tag{3}$$

Equations of mass diffusion

$$\frac{\partial S_1}{\partial t} + u_j \frac{\partial S_1}{\partial x_j} = \kappa_1 \nabla^2 S_1, \tag{4}$$

$$\frac{\partial S_2}{\partial t} + u_j \frac{\partial S_2}{\partial x_j} = \kappa_2 \nabla^2 S_2. \tag{5}$$

Equation of state

$$\rho = \rho_0[1 + \alpha(T_0 - T) - \alpha'(S_{10} - S_1) - \alpha''(S_{20} - S_2)], \tag{6}$$

where

$$\delta\rho = -\rho_0 \alpha(T - T_0), \tag{7}$$

$$\delta\rho' = \rho_0 \alpha'(S_1 - S_{10}), \tag{8}$$

$$\delta\rho'' = \rho_0 \alpha''(S_2 - S_{20}). \tag{9}$$

In the above equations ρ is the density, t is the time, $x_j (j = 1, 2, 3)$ are the Cartesian coordinates, $u_j (j = 1, 2, 3)$ are the velocity components, $X_i (i = 1, 2, 3)$ are the components of the external force, $\frac{P_1}{\rho_0} - \frac{1}{2}\left|\vec{\Omega} \times \vec{r}\right|^2$ is the hydrostatic pressure, $\vec{\Omega}$ is the angular velocity, $\vec{r}$ is the position vector, ν is the kinematic viscosity, T is the temperature, S_1 and S_2 are the two concentrations. κ is the thermal diffusivity, κ_1 and κ_2 are the coefficients of the mass diffusivity of two concentrations components respectively with $\kappa_1 > \kappa_2$, α, α' and α'' are respectively the coefficients of volume expansion due to temperature variation and concentration variation for the two concentration components, and ρ_0 is the density at some properly chosen mean temperature T_0 and concentrations S_{10} and S_{20}.

Now the initial state solution on the basis of initial state

$$(u, v, w) \equiv (0, 0, 0), \quad P_1 \equiv P_1(z), \qquad T \equiv T(z), \qquad S_1 \equiv S_1(z),\ S_2 \equiv S_2(z)$$

and $\rho \equiv \rho(z)$ is given by

$$(u, v, w) \equiv (0, 0, 0), T = T_0 - \beta z,\ S_1 = S_{10} - \beta' z, S_2 = S_{20} - \beta'' z,$$

$$\rho = \rho_0[1 + (\alpha\beta - \alpha'\beta' - \alpha''\beta'')z],$$

$$\frac{P_1}{\rho_0} - \frac{1}{2}\left|\vec{\Omega} \times \vec{r}\right|^2 = P = P_0 - g\rho_0\left[z + (\alpha\beta - \alpha'\beta' - \alpha''\beta'')\frac{z^2}{2}\right], \tag{10}$$

where P_0 is the pressure at the lower boundary $z = 0$, $\beta = \frac{T_0 - T_1}{d}$ is the maintained uniform adverse temperature gradient, $\beta' = \frac{S_{10} - S_{11}}{d}$ and $\beta'' = \frac{S_{20} - S_{21}}{d}$ are the maintained uniform non-adverse concentration gradients.

To study the stability of the system, we perturb all the variables in the form

$$(\bar{u}, \bar{v}, \bar{w}) = (0 + u', 0 + v', 0 + w'), \bar{T} = T_0 - \beta z + \theta', \bar{S}_1 = S_{10} - \beta' z + \phi_1',$$

$$\bar{S}_2 = S_{20} - \beta'' z + \phi_2', \quad \bar{\rho} = \rho_0[1 + \alpha(T_0 - T - \theta') - \alpha'(S_{10} - S_1 - \phi_1')$$
$$-\alpha''(S_{20} - S_2 - \phi_2')], \ \bar{P} = P_0 - g\rho_0\left[z + (\alpha\beta - \alpha'\beta' - \alpha''\beta'')\frac{z^2}{2}\right] + \delta P', \quad (11)$$

where $u', v', w', \theta', \phi_1', \phi_2', \delta P'$ are the perturbed variables and are assumed to be small.

Substituting (11) into Equations (1) – (5), we obtain the following linearized perturbation equations

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad (12)$$

$$\frac{\partial u'}{\partial t} = -\frac{\partial(\delta P')}{\partial x} + \nu\nabla^2 u' + 2\Omega v', \quad (13)$$

$$\frac{\partial v'}{\partial t} = -\frac{\partial(\delta P')}{\partial y} + \nu\nabla^2 v' - 2\Omega u', \quad (14)$$

$$\frac{\partial w'}{\partial t} = -\frac{\partial(\delta P')}{\partial z} + \nu\nabla^2 w' + g\alpha\theta' - g\alpha'\phi_1' - g\alpha''\phi_2', \quad (15)$$

$$\frac{\partial \theta'}{\partial t} - \beta w' = \kappa_0\nabla^2\theta', \quad (16)$$

$$\frac{\partial \phi_1'}{\partial t} - \beta' w' = \kappa_{10}\nabla^2\phi_1', \quad (17)$$

and

$$\frac{\partial \phi_2'}{\partial t} - \beta'' w' = \kappa_{20}\nabla^2\phi_2'. \quad (18)$$

The normal mode expansions of the dependent variables $u', v', w', \theta', \phi_1', \phi_2'$ and $\delta P'$ are assumed in the form

$$F'(x, y, z, t) = F''(z)\exp\left[i\left(k_x x + k_y y\right) + nt\right], \quad (19)$$

where $k = \sqrt{{k_x}^2 + {k_y}^2}$ is the wave number of perturbation k_x and k_y being real constants and *n* is a constant which can be complex in general. For functions with this dependence on *x*, *y* and *t* we have

$$\frac{\partial}{\partial t} = n, \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2 \text{ and } \nabla^2 = \frac{d^2}{dz^2} - k^2. \quad (20)$$

Equations (12) – (18), then give

$$ik_x u'' + ik_y v'' + \frac{dw''}{dz} = 0, \quad (21)$$

$$nu'' = -ik_x(\delta P'') + \nu\left(\frac{d^2}{dz^2} - k^2\right)u'' + 2\Omega v'', \quad (22)$$

$$nv'' = -ik_y(\delta P'') + \nu\left(\frac{d^2}{dz^2} - k^2\right)v'' - 2\Omega u'', \quad (23)$$

$$nw'' = -\frac{d(\delta P'')}{dz} + \nu\left(\frac{d^2}{dz^2} - k^2\right)w'' + g\alpha\theta'' - g\alpha'\phi_1'' - g\alpha''\phi_2'', \tag{24}$$

$$n\theta'' - \beta w'' = \kappa_0\left(\frac{d^2}{dz^2} - k^2\right)\theta'', \tag{25}$$

$$n\phi_1{}'' - \beta' w'' = \kappa_{10}\left(\frac{d^2}{dz^2} - k^2\right)\phi_1'', \tag{26}$$

and

$$n\phi_2{}'' - \beta'' w'' = \kappa_{20}\left(\frac{d^2}{dz^2} - k^2\right)\phi_2''. \tag{27}$$

Eliminating u'' and v'' from the left hand side of Equations (22) and (23) by multiplying Equation (22) by ik_x and Equation (23) by ik_y respectively, adding the resulting equations and using Equation (21); and then eliminating $\delta P''$between this resulting equation and Equation (24), we have the equation

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu}\right)w'' = \frac{g\alpha k^2\theta''}{\nu} - \frac{g\alpha' k^2\phi_1''}{\nu} - \frac{g\alpha'' k^2\phi_2''}{\nu} + \frac{2\Omega}{\nu}\frac{d\zeta''}{dz}, \tag{28}$$

where

$$\zeta'' = i\left(k_x v'' - k_y u''\right) \tag{29}$$

is the z component of vorticity.

Further, Equations (25) – (27) can be written as

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\kappa_0}\right)\theta'' = -\frac{\beta}{\kappa_0}w'', \tag{30}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\kappa_{10}}\right)\phi_1'' = -\frac{\beta'}{\kappa_{10}}w'', \tag{31}$$

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\kappa_{20}}\right)\phi_2'' = -\frac{\beta'}{\kappa_{20}}w'', \tag{32}$$

respectively.

In order to obtain an equation governing ζ'',we multiply Equation (22) by k_y and Equation (23) by k_x subtract the former resulting equation from the latter resulting equation and then using Equation (21) to obtain

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{n}{\nu}\right)\zeta'' = -\frac{2\Omega}{\nu}\frac{dw''}{dz}. \tag{33}$$

Now by introducing the non-dimensional quantities defined by

$$z_* = \frac{z}{d}, \tau_{1*} = \frac{\kappa_{10}}{\kappa_0}, \tau_{2*} = \frac{\kappa_{20}}{\kappa_0}, \sigma_* = \frac{\nu}{\kappa_0}, D_* = d\frac{d}{dz},\ p_* = \frac{nd^2}{\kappa_0}, a_* = kd, R_* = \frac{g\alpha\beta d^4}{\kappa\nu},$$

$$R_{1*} = \frac{g\alpha'\beta' d^4}{\kappa\nu},\ \ R_{2*} = \frac{g\alpha''\beta'' d^4}{\kappa\nu}, T_{a_*} = \frac{4\Omega^2 d^4}{\nu^2}, w_* = \frac{\beta d^2 w''}{\kappa_0}, \theta_* = \theta'', \phi_{1*} = \frac{\beta\phi_1{}''}{\beta'},$$

$$\phi_{2*} = \frac{\beta\phi_2{}''}{\beta''},\ \zeta_* = \frac{\beta\nu d\zeta''}{2\Omega\kappa_0}, \tag{34}$$

we can reduce Equations (28), (30) – (33) to the following non dimensional forms (omitting the asterisks for simplicity in writing):

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right)w = Ra^2\theta - R_1a^2\phi_1 - R_2a^2\phi_2 + T_aD\zeta, \tag{35}$$

$$(D^2 - a^2 - p)\theta = -w, \tag{36}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{37}$$

$$\left(D^2 - a^2 - \frac{p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}, \tag{38}$$

$$\left(D^2 - a^2 - \frac{p}{\sigma}\right)\zeta = -Dw. \tag{39}$$

Equations (35) – (39) are to be solved using the following appropriate boundary conditions:

$$w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta \text{ at } z = 0 \text{ and } z = 1, \tag{40}$$

(Both the boundaries are rigid)

or

$$w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta \text{ at } z = 0 \text{ and } z = 1, \tag{41}$$

(both the boundaries are free)

or

$$w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta \text{ at } z = 0,$$

(lower boundary is rigid)

and

$$w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta \text{ at } z = 1, \tag{42}$$

(upper boundary is free)

or

$$w = 0 = \theta = \phi_1 = \phi_2 = D^2w = D\zeta \text{ at } z = 0,$$

(lower boundary is free)

and

$$w = 0 = \theta = \phi_1 = \phi_2 = Dw = \zeta \text{ at } z = 1, \tag{43}$$

(upper boundary is rigid)

where z is the real independent variable such that $0 \leq z \leq 1$, $D = \frac{d}{dz}$ is the differentiation along the vertical coordinate, $a^2 > 0$ is a constant, $\sigma > 0$ is a constant, $\tau_1 > 0$ is a constant, $\tau_2 > 0$ is a constant, $R > 0$, $R_1 > 0, R_2 > 0$ are constants, $T_a > 0$ is a constant, $p = p_r + ip_i$ is a complex constant such that p_r and p_i are real constants and as a consequence the dependent variables

$$w(z) = w_r(z) + iw_i(z), \theta(z) = \theta_r(z) + i\theta_i(z), \phi_1(z) = \phi_{1r}(z) + i\phi_{1i}(z),$$

and

$$\phi_2(z) = \phi_{2r}(z) + i\phi_{2i}(z), \zeta(z) = \zeta_r(z) + i\zeta_i(z)$$

are complex valued functions of the real variable z such that

$$w_r(z), w_i(z), \theta_r(z), \theta_i(z), \phi_{1r}(z), \phi_{1i}(z), \phi_{2r}(z), \phi_{2i}(z), \zeta_r(z) \text{ and } \zeta_i(z)$$

are real valued functions of the real variable z. The meaning of the symbols from the physical point of view are as follows: z is the vertical coordinate, $D = \frac{d}{dz}$ is the differentiation along vertical direction, a^2 is the square of wave number, $\sigma = \frac{\nu}{\kappa}$ is the Prandtl number, $\tau_1 = \frac{\kappa_1}{\kappa}$ and $\tau_2 = \frac{\kappa_2}{\kappa}$ are the Lewis numbers for the two concentration components with mass diffusivities κ_1 and κ_2 respectively and κ is thermal diffusivity, T_a is the Taylor number, R is the Rayleigh number, R_1 and R_2 are concentration Rayleigh numbers for the two concentration components, p is the complex growth rate, w is the vertical velocity, θ is the temperature, ϕ_1, ϕ_2 are the two concentrations and ζ is the vertical vorticity. It may further be noted that Equations (35) – (43) describe an eigenvalue problem for p and govern rotatory triply diffusive convection for any combination of dynamically free and rigid boundaries.

We prove the following theorem:

Theorem 1.

If $R > 0, R_1 > 0, R_2 > 0, T_a > 0, p_r \geq 0,\ p_{\mathrm{i}} \neq 0,$ then a necessary condition for the existence of nontrivial solution $(w,\ \theta, \phi_1, \phi_2, \zeta, p)$ of Equations (35) - (39) together with boundary conditions (40) - (43) is that

$$|p|^2 < \max\{(R_1 + R_2)\sigma, T_a\sigma^2\}.$$

Proof:

Multiplying Equation (35) by w^* (the superscript $*$ denotes the complex conjugation) throughout, and integrating the resulting equation over the vertical range of z, we get

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w dz$$
$$= Ra^2 \int_0^1 w^*\theta dz - R_1 a^2 \int_0^1 w^*\phi_1 dz - R_2 a^2 \int_0^1 w^*\phi_2 dz + T_a \int_0^1 w^* D\zeta dz. \quad (44)$$

Making use of Equations (36) – (39) and the fact that $w(0) = 0 = w(1)$, we can write

$$Ra^2 \int_0^1 w^*\theta dz = -Ra^2 \int_0^1 \theta(D^2 - a^2 - p^*)\theta^* dz, \quad (45)$$

$$-R_1 a^2 \int_0^1 w^*\phi_1 dz = R_1 a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right) \phi_1{}^* dz, \quad (46)$$

$$-R_2 a^2 \int_0^1 w^*\phi_2 dz = R_2 a^2 \tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{p^*}{\tau_2}\right) \phi_2{}^* dz, \quad (47)$$

$$T_a \int_0^1 w^* D\zeta dz = -T_a \int_0^1 \zeta D w^* dz = T_a \int_0^1 \zeta \left(D^2 - a^2 - \frac{p^*}{\sigma}\right) \zeta^* dz. \tag{48}$$

Combining Equations (44) – (48), we get

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w dz$$

$$\int_0^1 w^*(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{\sigma}\right) w dz$$

$$= -Ra^2 \int_0^1 \theta(D^2 - a^2 - p^*)\theta^* dz$$

$$+R_1 a^2 \tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right) {\phi_1}^* dz$$

$$+R_2 a^2 \tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{p^*}{\tau_2}\right) {\phi_2}^* dz$$

$$+T_a \int_0^1 \zeta \left(D^2 - a^2 - \frac{p^*}{\sigma}\right) \zeta^* dz. \tag{49}$$

Integrating the various terms of Equation (49), by parts, for an appropriate number of times and utilizing either of the boundary conditions (40) – (43) it follows that

$$\int_0^1 (|D^2 w|^2 + 2a^2 |Dw|^2 + a^4 |w|^2) dz + \frac{p}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2)\, dz$$

$$= Ra^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2 + p^* |\theta|^2)\, dz$$

$$-R_1 a^2 \tau_1 \int_0^1 \left(|D\phi_1|^2 + a^2 |\phi_1|^2 + \frac{p^*}{\tau_1} |\phi_1|^2\right) dz$$

$$-R_2 a^2 \tau_2 \int_0^1 \left(|D\phi_2|^2 + a^2 |\phi_2|^2 + \frac{p^*}{\tau_2} |\phi_2|^2\right) dz$$

$$-T_a \int_0^1 \left(|D\zeta|^2 + a^2 |\zeta|^2 + \frac{p^*}{\sigma} |\zeta|^2\right) dz. \tag{50}$$

Equating the imaginary parts of both sides of Equation (50) and cancelling $p_i (\neq 0)$ throughout from the resulting equation, we obtain

$$\frac{1}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz = -Ra^2 \int_0^1 |\theta|^2 dz + R_1 a^2 \int_0^1 |\phi_1|^2\, dz$$

$$+R_2 a^2 \int_0^1 |\phi_2|^2\, dz + \frac{T_a}{\sigma} \int_0^1 |\zeta|^2 dz. \tag{51}$$

Now, from Equation (37), we derive that

$$\int_0^1 \left(D^2 - a^2 - \frac{p}{\tau_1}\right)\phi_1 . \left(D^2 - a^2 - \frac{p^*}{\tau_1}\right){\phi_1}^* dz = \frac{1}{{\tau_1}^2}\int_0^1 |w|^2 dz. \tag{52}$$

Integrating the various terms on the left hand side of Equation (52) by parts for an appropriate number of times and making use of the boundary conditions on ϕ_1 namely $\phi_1(0) = 0 = \phi_1(1)$, it follows that

$$\int_0^1 (|D^2\phi_1|^2 + 2a^2|D\phi_1|^2 + a^4|\phi_1|^2)dz$$
$$+ \frac{2p_r}{\tau_1}\int_0^1 (|D\phi_1|^2 + a^2|\phi_1|^2)\, dz + \frac{|p|^2}{{\tau_1}^2}\int_0^1 |\phi_1|^2 dz = \frac{1}{{\tau_1}^2}\int_0^1 |w|^2 dz. \tag{53}$$

Since $p_r \geq 0$, it follows from Equation (53), that

$$\int_0^1 |\phi_1|^2 dz < \frac{1}{|p|^2}\int_0^1 |w|^2 dz. \tag{54}$$

In the same manner, from Equations (38) and (39), we respectively have

$$\int_0^1 \left(\left|D^2\phi_2\right|^2 + 2a^2\left|D\phi_2\right|^2 + a^4\left|\phi_2\right|^2\right) dz$$
$$+ \frac{2p_r}{\tau_2}\int_0^1 (|D\phi_2|^2 + a^2|\phi_2|^2)\, dz + \frac{|p|^2}{{\tau_2}^2}\int_0^1 |\phi_2|^2 dz = \frac{1}{{\tau_2}^2}\int_0^1 |w|^2 dz, \tag{55}$$

and

$$\int_0^1 (|D^2\zeta|^2 + 2a^2|D\zeta|^2 + a^4|\zeta|^2)dz$$
$$+ \frac{2p_r}{\sigma}\int_0^1 (|D\zeta|^2 + a^2|\zeta|^2)\, dz + \frac{|p|^2}{\sigma^2}\int_0^1 |\zeta|^2 dz = \int_0^1 |Dw|^2 dz. \tag{56}$$

Since $p_r \geq 0$, Equations (55) and (56) respectively gives

$$\int_0^1 |\phi_2|^2 dz < \frac{1}{|p|^2}\int_0^1 |w|^2 dz, \tag{57}$$

$$\int_0^1 |\zeta|^2 dz \leq \frac{\sigma^2}{|p|^2}\int_0^1 |Dw|^2 dz. \tag{58}$$

Now, making use of inequalities (54), (57) and (58), we can write Equation (51) as

$$\frac{1}{\sigma}\left(1 - \frac{T_a\sigma^2}{|p|^2}\right)\int_0^1 |Dw|^2 dz + \frac{a^2}{\sigma}\left[1 - \frac{(R_1+R_2)\sigma}{|p|^2}\right]\int_0^1 |w|^2\, dz + Ra^2\int_0^1 |\theta|^2 dz < 0, \tag{59}$$

which clearly implies that

$$|p|^2 < \max\{(R_1 + R_2)\sigma, T_a\sigma^2\}. \tag{60}$$

This establishes the theorem.

Theorem 1, from the physical point of view, states that the complex growth rate (p_r, p_i) of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude, in a rotatory triply diffusive fluid layer with one of the components as heat with diffusivity κ, must lie inside a semicircle in the right half of the $(p_r, p_i) -$ plane whose centre is the origin and $(\text{radius})^2 = \max.\{(R_1 + R_2)\sigma, T_a\sigma^2\}$. A general plot showing the region of complex growth rate is given below.

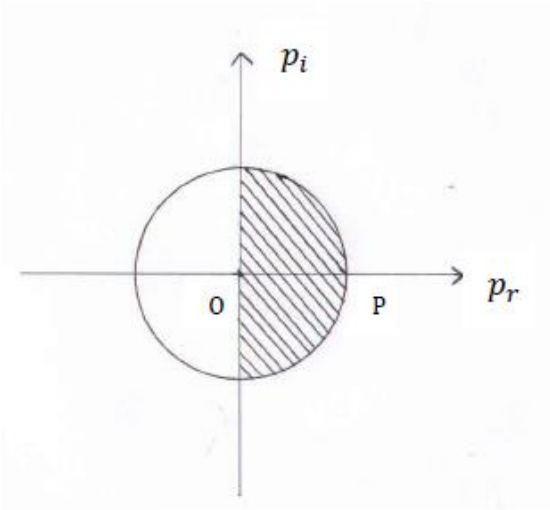

Figure 2. Shaded region shows the region of complex growth rate $(OP)^2 = max.\{(R_1 + R_2)\sigma, T_a\sigma^2\}$

3. Special Cases:

The following results may be obtained from Theorem 1 as special cases:

1. For rotatory Rayleigh-Benard convection $(R_1 = 0 = R_2, T_a > 0, |p|^2 < T_a\sigma^2$. (Banerjee et al. (1981)).

2. For thermohaline convection of Veronis type (1965) $(R_1 > 0, R_2 = 0 = T_a)$, $|p|^2 < R_1\sigma$. (Banerjee et al. (1981)).

3. For rotatory thermohaline convection Veronis type (1965) $(R_1 > 0, R_2 = 0, T_a > 0)$, $|p|^2 < \max.\{R_1\sigma, T_a\sigma^2\}$. (Gupta et al. (1983)).

4. For thermohaline convection of Stern type (1960) $(R_1 < 0, R_2 = 0 = T_a)$, $|p|^2 < |R|\sigma$. (Banerjee et al. (1981)).

5. For rotatory thermohaline convection of Stern type (1960) $(R_1 < 0, R_2 = 0, T_a > 0)$, $|p|^2 < \max.\{|R|\sigma, T_a\sigma^2\}$. (Gupta et al. (1983)).

6. For rotatory triply diffusive convection analogous to rotatory thermohaline convection of Stern type (1960) $(R_1 < 0, R_2 < 0, T_a > 0)$, $|p|^2 < \max.\{|R|\sigma, T_a\sigma^2\}$.

Proof:

Putting $R_1 = -|R_1|$and $R_2 = -|R_2|$ in Equation (35), and adopting the same procedure as is used to prove Theorem 1, we obtain the desired result.

7. For triply diffusive convection analogous to thermohaline convection of Stern type (1960) ($R_1 < 0, R_2 < 0, T_a = 0$), $|p|^2 < |R|\sigma$.

4. Conclusions

Linear stability theory is used to investigate triply diffusive convection in the presence of uniform vertical rotation. Upper bounds for the complex growth rate of an arbitrary neutral or unstable oscillatory perturbation of growing amplitude are obtained. These bounds are important especially when both the boundaries are not dynamically free so that exact solutions in closed form are not obtainable. It is further proved that the results obtained herein are uniformly valid for the quite general nature of the bounding surfaces.

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