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Reduction of a nilpotent intuitionistic fuzzy matrix using implication operator

Riyaz Ahmad Padder & P. Murugadas

Department of Mathematics
Annamalai University
Annamalainagar - 608 002, India
padderriyaz01@gmail.com; bodi_muruga@yahoo.com

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Abstract

A problem of reducing intuitionistic fuzzy matrices is examined and some useful properties are obtained with respect to nilpotent intutionistic fuzzy matrices. First, reduction of irreflexive and transitive intuitionistic fuzzy matrices are considered, and then the properties are applied to nilpotent intutionistic fuzzy matrices. Nilpotent intuitionistic fuzzy matrices are intuitionistic fuzzy matrices which signify acyclic graphs, and the graphs are used to characterize consistent systems. The properties are handy for generalization of various systems with intuitionistic fuzzy transitivity.

Keywords: Intuitionistic fuzzy sets; Intuitionistic fuzzy matrix; Intuitionistic fuzzy transitive matrix; Intuitionistic fuzzy irreflexive matrix; Intuitionistic fuzzy nilpotent matrix

MSC 2010 No.: 03E72, 15B15

1. Introduction

Many theories have been put forth over the years to deal with the various types of uncertainties. These theories are put into practice and when found to be wanting are improved upon, paving the way for new theories to handle the tricky uncertainties. In 1965, Zadeh (1965) came out with the concept of fuzzy set which is indeed an extension of the classical notion of set. However, it often falls short of the expected standard when describing the neutral state. As a result, a new concept namely Intuitionistic Fuzzy Set (IFS) was introduced by Atanassov (1983) and represented it as $A = [\langle x, \mu_A(x), \nu_A(x)\rangle / x \in E$, where E denotes a universal set] in which $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ denote membership and non-membership functions of A, respectively, and its sum is less than or equal to one. In short we write the elements of IFS as $\langle x, x'\rangle$ such that $x + x' \leq 1$. The ideas of IFS were developed later in Atanassov (1998, 1999, 2005a, 2005b, 2005c). Xu and Yager (2006) developed some geometric operators based on IFSs.

The notion Intuitionistic Fuzzy Matrix (IFM) was introduced by Atanssov (1987). The index matrix representation of the intuitionistic fuzzy graphs has been studied in Atanssov (1994). Pal et al. (2002), Meenakshi and Gandhimathi (2010), and Sriram and Murugadas (2011, 2010) studied IFM for finding intuitionistic fuzzy linear relation equation, g-inverse and intuitionistic fuzzy linear transformation and others. Meenakshi (2008) studied minus ordering, space ordering and schur complement of fuzzy matrix and block fuzzy matrix. Shyamal and Pal (2002) have studied the distance between IFM. Bhowmik and Pal (2008a, 2008b) examined circulant IFM and generalized IFMs. Im (2006) studied the determinant of square IFMs. In (2014) Atanassov studied intuitionistic fuzzy index matrix on the basis of index matrix and extended intuitionistic fuzzy index matrix. Several authors (Adak et al. (2011, 2012), Lee and Jeong (2005), Mondal and Pal (2013, 2014), Murugadas and Padder (2015, 2016a, 2016b), Pradhan and Pal (2012, 2013, 2014)) worked on IFMs and obtained various interesting results, which are very useful in handling uncertainty problems in our daily life.

Hashimoto (1982a, 1984, 2005) used implication operators in fuzzy matrices and studied some properties. Murugadas (2011) and Murugadas and Lalitha (2012, 2014a, 2014b, 2014c, 2016) used implication operators in IFM for obtaining g-inverse, sub-inverse and decomposition of an IFM. Tan (2005) obtained some important properties of reduction of nilpotent fuzzy matrix. Lur et al. (2003) studied simultaneously nilpotent fuzzy matrices. Lur et al. (2004) obtained some properties of nilpotent matrices in terms of eigenvalues. Han et al. (2005) studied some important properties of the reduction of nilpotent incline matrices over a additively residuated incline. Hashimoto (1982b) studied the reduction of fuzzy matrices and obtained some results of nilpotent fuzzy matrices. The purpose of this paper is to study the reduction of nilpotent IFM by using the implication operator.

2. Preliminaries

Throughout the paper, matrix means IFM. Atanassov introduced the following operations on IFS for $\langle x, x'\rangle, \langle y, y'\rangle \in IFS$, $\langle x, x'\rangle \vee \langle y, y'\rangle = \langle max\{x, y\}, min\{x', y'\}\rangle$ and $\langle x, x'\rangle \wedge \langle y, y'\rangle = \langle min\{x, y\}, max\{x', y'\}\rangle$. $\langle x, x'\rangle \geq \langle y, y'\rangle \Rightarrow x \geq y$ and $x' \leq y'$, therefore in this case we say $\langle x, x'\rangle$ and $\langle y, y'\rangle$ are comparable.

For any two comparable elements $\langle x, x'\rangle$, $\langle y, y'\rangle \in$ IFS, the operation $\langle x, x'\rangle \leftharpoondown \langle y, y'\rangle$ is defined as

$$\langle x, x' \rangle \leftharpoondown \langle y, y' \rangle = \begin{cases} \langle x, x' \rangle & \text{if } \langle x, x' \rangle > \langle y, y' \rangle, \\ \langle 0, 1 \rangle & \text{if } \langle x, x' \rangle \leq \langle y, y' \rangle. \end{cases}$$

Definition 2.1. (Pal et al. (2002))

Let $X = \{x_1, x_2, ...x_m\}$ be a set of alternatives and $Y = \{y_1, y_2, ...y_n\}$ be the attribute set of each element of X. An Intuitionistic Fuzzy Matrix (IFM) is defined by

$$A = (\langle (x_i, y_j), \mu_A(x_i, y_j), \nu_A(x_i, y_j) \rangle),$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$, where $\mu_A : X \times Y \to [0, 1]$ and $\nu_A : X \times Y \to [0, 1]$ satisfies the condition $0 \leq \mu_A(x_i, y_j) + \nu_A(x_i, y_j) \leq 1$. For simplicity we denote an intuitionistic fuzzy matrix (IFM) is a matrix of pairs $A = (\langle a_{ij}, a'_{ij} \rangle)$ of a non negative real numbers satisfying $a_{ij} + a'_{ij} \leq 1$ for all i, j. We denote the set of all IFM of order $m \times n$ by $\mathscr{F}_{mn}$.

Some usual matrix operations on IFMs are listed below.

For $n \times n$ IFMs $Q = (\langle q_{ij}, q'_{ij} \rangle)$ and $S = (\langle s_{ij}, s'_{ij} \rangle)$ some operations are defined as follows:

$Q \vee S = (\langle q_{ij} \vee s_{ij}, q'_{ij} \wedge s'_{ij} \rangle)$,
$Q \wedge S = (\langle q_{ij} \wedge s_{ij}, q'_{ij} \vee s'_{ij} \rangle)$,
$Q \overset{c}{\leftharpoondown} S = (\langle q_{ij}, q'_{ij} \rangle \overset{c}{\leftharpoondown} \langle s'_{ij}, s'_{ij} \rangle)$ (component wise),
$Q \times S = (\langle q_{i1} \wedge s_{1j}, q'_{i1} \vee s'_{1j} \rangle \vee \langle q_{i2} \wedge s_{2j}, q'_{i2} \vee s'_{2j} \rangle \vee \ldots \vee \langle q_{in} \wedge s_{nj}, q'_{in} \vee s'_{nj} \rangle)$,
$Q/S = Q \overset{c}{\leftharpoondown} (Q \times S)$,
$Q^1 = Q$,
$Q^{k+1} = Q^k \times Q, \quad k = 1, 2, 3...$,
$Q^+ = Q \vee Q^2 \vee ... \vee Q^n$, and
$Q \leq S (S \geq Q)$ if and only if $\langle q_{ij}, q'_{ij} \rangle \leq \langle s_{ij}, s'_{ij} \rangle$.

A zero matrix is a matrix whose entries are $\langle 0, 1 \rangle$.

An IFM Q is called transitive if $Q^2 \leq Q$. An IFM Q is said to be irreflexive if all of its diagonal elements are zero i.e $\langle 0, 1 \rangle$. An IFM Q is said to be nilpotent if there exists a natural number n, such that $Q^n = (\langle 0, 1 \rangle)$.

The transitive closure of a matrix Q is Q^+ and closure properties are familiar. A transitive matrix Q is a matrix which signifies a transitive relation. Transitive relations are used in several applications (Ovchinnikov (1981), Zadeh (1971)).

If a matrix is irreflexive and transitive the matrix is nilpotent, which implies if Q is nilpotent then Q is irreflexive and there is a permutation matrix P such that $P \times Q \times P^T$ is an upper triangular or lower triangular with $\langle 0, 1 \rangle$ diagonal elements, where P^T is the transpose of P.

From definition $Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q)$, the (i,j) entry of Q/Q is either $\langle q_{ij}, q'_{ij} \rangle$ or $\langle 0, 1 \rangle$.

3. Reduction of Intuitionistic Fuzzy Nilpotent Matrix (IFNM)

During the present study, we examine the reduction of irreflexive and transitive IFMs and obtain some theorems. Then we use these theorems to nilpotent IFMs in order to obtain some properties of reduction of nilpotent IFMs.

Theorem 3.1.

If Q is an $n \times n$ irreflexive and transitive IFM, then

$$(Q/Q)^+ = Q.$$

Proof:

Since $S = (\langle s_{ij}, s'_{ij}\rangle) = Q/Q$, $S^k = (\langle s^{(k)}_{ij}, s'^{(k)}_{ij}\rangle)$. Let Q and S are nilpotent.

Since, clearly,

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee ... \vee (Q/Q)^{n-1} \leq Q,$$

we have to prove that

$$Q \leq (Q/Q) \vee (Q/Q)^2 \vee ... \vee (Q/Q)^{n-1}.$$

That is, $\langle q_{ij}, q'_{ij}\rangle \leq \langle s^{(k)}_{ij}, s'^{(k)}_{ij}\rangle$ for some $k(1 \leq k \leq n-1)$.

Suppose that $\langle q_{ij}, q'_{ij}\rangle \geq \langle s^{(k)}_{ij}, s'^{(k)}_{ij}\rangle$ for every $k = 1, 2, ..., n-1$.

(1) (a) Since $\langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle s_{ij}, s'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{il} \wedge q_{lj}), \bigwedge_{k=1}^{n} (q'_{il} \vee q'_{lj})\rangle = \langle 0, 1\rangle.$$

That is,

$$\langle q_{ij}, q'_{ij}\rangle \leq \langle \bigvee_{l=1}^{n} (q_{il} \wedge q_{lj}), \bigwedge_{k=1}^{n} (q'_{il} \vee q'_{lj})\rangle.$$

Thus, we can find l_{11} such that $\langle q_{il_{11}}, q'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{l_{11}i}, q'_{l_{11}i}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

It follows that $\langle q^{(2)}_{ij}, q'^{(2)}_{ij}\rangle \geq \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$. Therefore, $\langle q^{(2)}_{ij}, q'^{(2)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle$, since Q is transitive.

We now prove that $\langle s_{il_{1_{p(1)}}}, s'_{il_{1_{p(1)}}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{l_{1_{p(1)}}}, q'_{l_{1_{p(1)}}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ for a few $l_{1_{p(1)}}$.

(b) If, $\langle s_{il_{11}}, s'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, then we put $p(1) = 1$.

If, $\langle s_{il_{11}}, s'_{il_{11}}\rangle = \langle 0,1\rangle$, that is, if

$$\langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{11}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{11}})\rangle = \langle 0,1\rangle,$$

then, $\langle q_{il_{11}}, q'_{il_{11}}\rangle \leq \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{11}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{11}})\rangle$.

Since $\langle q_{il_{11}}, q'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,\ \langle q_{l_{12}l_{11}}, q'_{l_{12}l_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle \text{ and}$$

$$\langle q^{(3)}_{ij}, q'^{(3)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0,1\rangle \text{ for a few } l_{12}.$$

Further, since

$$(\langle q_{l_{12}l_{11}}\wedge q_{l_{11}j}, q'_{l_{12}l_{11}}\vee q'_{l_{11}j}\rangle) \leq \langle q_{l_{12}j}, q'_{l_{12}j}\rangle \text{ and } \langle q_{l_{11}j}, q'_{l_{11}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

we have $\langle q_{l_{12}j}, q'_{l_{12}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

(c) Moreover, if $\langle s_{il_{11}}, s'_{il_{11}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, then we put $p(1) = 2$.

If, $\langle s_{il_{12}}, s'_{il_{12}}\rangle = \langle 0,1\rangle$ that is, $\langle q_{il_{12}}, q'_{il_{12}}\rangle \overset{c}{\leftharpoondown} \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{12}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{12}})\rangle = \langle 0,1\rangle$,

then,

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \leq \langle \bigvee_{k=1}^{n}(q_{il}\wedge q_{ll_{12}}), \bigwedge_{k=1}^{n}(q'_{il}\vee q'_{ll_{12}})\rangle = \langle 0,1\rangle.$$

Since $\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, we get

$$\langle q_{il_{12}}, q'_{il_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,\ \langle q_{l_{13}l_{12}}, q'_{l_{13}l_{12}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle \text{ and}$$

$$\langle q^{(4)}_{ij}, q'^{(4)}_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0,1\rangle \text{ for some } l_{13}.$$

Thus, since

$$(\langle q_{l_{13}l_{12}}\wedge q_{l_{12}j}, q'_{l_{13}l_{12}}\vee q'_{l_{12}j}\rangle) \leq \langle q_{l_{13}j}, q'_{l_{13}j}\rangle \text{ and } \langle q_{l_{12}j}, q'_{l_{12}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

we have $\langle q_{l_{13}j}, q'_{l_{13}j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

(d) By repeating the same process, since Q is nilpotent for a few $l_{ip(1)}$ such that $p(1) < n-1$, we get

$$\langle s_{il_{1p(1)}}, s'_{il_{1p(1)}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \langle q_{il_{1p(1)}}, q'_{il_{1p(1)}}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

(2) Next, since $\langle s_{ij}^{(2)}, s_{ij}^{\prime(2)}\rangle < \langle q_{ij}, q_{ij}^{\prime}\rangle$, we get $\langle s_{l_{1p(1)j}}, s_{l_{1p(1)j}}^{\prime}\rangle < \langle q_{ij}, q_{ij}^{\prime}\rangle$.

Then, by $\langle q_{l_{1p(1)j}}, q_{il_{1p(1)j}}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle$ it follows that

$$\langle q_{l_{1p(1)j}}, q_{l_{1p(1)j}}^{\prime}\rangle \leftharpoonup \langle \bigvee_{k=1}^{n}(q_{l_{1p(1)}l} \wedge q_{ij}), \bigwedge_{k=1}^{n}(q_{l_{1p(1)}l}^{\prime} \vee q_{ij}^{\prime})\rangle = \langle 0, 1\rangle,$$

that is,

$$\langle q_{l_{1p(1)j}}, q_{l_{1p(1)j}}^{\prime}\rangle \leq \langle \bigvee_{k=1}^{n}(q_{l_{1p(1)}l} \wedge q_{lj}), \bigwedge_{k=1}^{n}(q_{l_{1p(1)}l}^{\prime} \vee q_{lj}^{\prime})\rangle.$$

Since $\langle q_{ij}, q_{ij}^{\prime}\rangle \leq \langle q_{l_{1p(1)j}}, q_{l_{1p(1)j}}^{\prime}\rangle$, we have

$$\langle q_{l_{1p(1)}l_{21}}, q_{l_{1p(1)}l_{21}}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \langle q_{l_{21}j}, q_{l_{21}j}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \text{ and}$$

$$\langle q_{l_{1p(1)j}}^{(2)}, q_{l_{1p(1)j}}^{\prime(2)}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle > \langle 0, 1\rangle \text{ for a few } l_{21}.$$

By the same process as in (1) it follows as below

$$\langle s_{l_{1p(1)}l_{2p(2)}}, s_{l_{1p(1)}l_{2p(2)}}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \langle q_{l_{2p(2)}j}, q_{l_{2p(2)}j}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \text{ and}$$

$$\langle s_{il_{2p(2)}}^{(2)}, s_{il_{2p(2)}}^{\prime(2)}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle > \langle 0, 1\rangle \text{ for some } l_2p(2).$$

(3) By continuing the same process as above,

$$\langle s_{l_{n-1p(n-1)}l_{np(n)}}, s_{l_{n-1p(n-1)}l_{np(n)}}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \langle q_{l_{np(n)}j}, q_{l_{np(n)}j}^{\prime}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle, \text{ and}$$

$$\langle s_{il_{np(n)}}^{(n)}, s_{il_{np(n)}}^{(n)}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle > \langle 0, 1\rangle.$$

So, a contradiction arises with S nilpotent.

Thus, we get $\langle s_{ij}^{(k)}, s_{ij}^{(k)}\rangle \geq \langle q_{ij}, q_{ij}^{\prime}\rangle$ for a few $k(1 \leq k \leq n-1)$, so that,

$$S^{+} = Q.$$

The following example shows that Q/Q is the reduced form of Q and it is enough if we calculate the transitive closure of Q/Q instead of calculating transitive closure of Q.

Example 3.1.

Let Q be the following irreflexive and transitive intuitionistic fuzzy matrix

$$Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \text{ (irreflexive)},$$

$$Q^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix} \text{ (Transitive } Q^2 \leq Q).$$

Now,

$$Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q),$$

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}.$$

Therefore,

$$Q = (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3. \tag{1}$$

Remark 3.2.

In the above theorem irreflexivity and transitivity are essential. If any one of the above two conditions fail, then the result will fail. It is illustrated through the following example.

Example 3.2.

Let $Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \end{pmatrix}$. Clearly Q is not irreflexive.

$Q^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.1, 0.7\rangle \end{pmatrix} \Rightarrow (Q^2 \leq Q)$. So Q is transitive.

Now,

$$Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q),$$

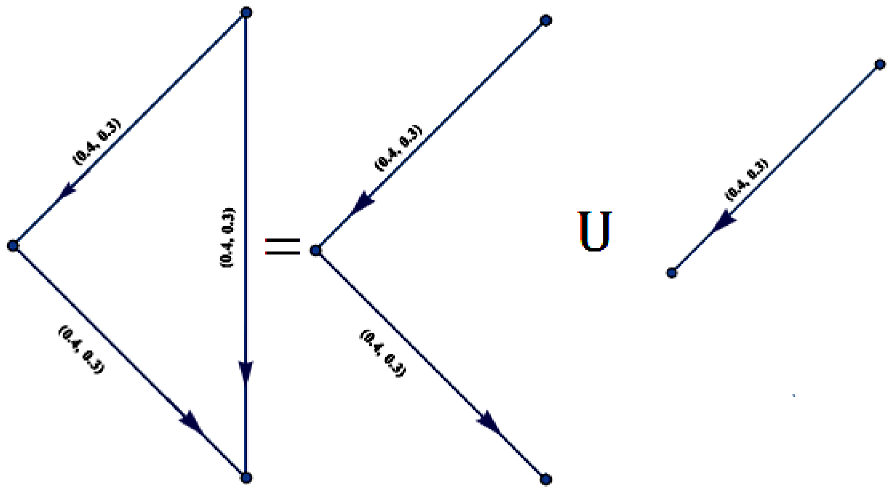

Figure 1: The graphical representation of equation (1)

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^2 = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}.$$

Therefore,

$$Q \neq (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2. \tag{2}$$

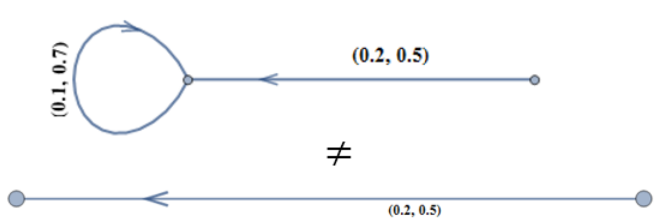

Figure 2: The graphical representation of equation (2)

Example 3.3.

Let

$Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle \end{pmatrix}$. Clearly Q is irreflexive.

$Q^2 = \begin{pmatrix} \langle 0.2, 0.5\rangle & \langle 0.2, 0.5\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.2, 0.5\rangle \end{pmatrix} \Rightarrow Q^2 \nleq Q$. So Q is not transitive.

Now,

$$Q/Q = Q \overset{c}{\leftharpoondown} (Q \times Q),$$

$$Q/Q = \begin{pmatrix} \langle 0.0, 1.0\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \end{pmatrix},$$

$$(Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3,$$

$$(Q/Q)^+ = \begin{pmatrix} \langle 0.2, 0.5\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle & \langle 0.0, 1.0\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.4, 0.3\rangle & \langle 0.2, 0.5\rangle \end{pmatrix}.$$

Therefore,

$$Q \neq (Q/Q)^+ = (Q/Q) \vee (Q/Q)^2 \vee (Q/Q)^3. \tag{3}$$

Theorem 3.3.

Let Q be an $n \times n$ irreflexive and transitive matrix. Then, the following conditions are equivalent:

(1) $Q/Q \leq S \leq Q$

(2) $S^+ = Q$ for any $n \times n$ IFM S.

Proof:

Let $S^k = (\langle s_{ij}^{(k)}, s_{ij}^{\prime(k)}\rangle)$ and $T = (\langle t_{ij}, t'_{ij}\rangle) = Q/Q$. That is,

$$\langle t_{ij}, t'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \overset{c}{\leftharpoondown} \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle.$$

(1) Suppose that $Q/Q \leq S \leq Q$. Clearly from Theorem 3.1, $S^+ = Q$.

(2) Suppose that $S^+ = Q$, then we get $S \leq Q$.

(a) Let $n = 1$.

The only irreflexive matrix is $\langle 0, 1\rangle$. Thus,

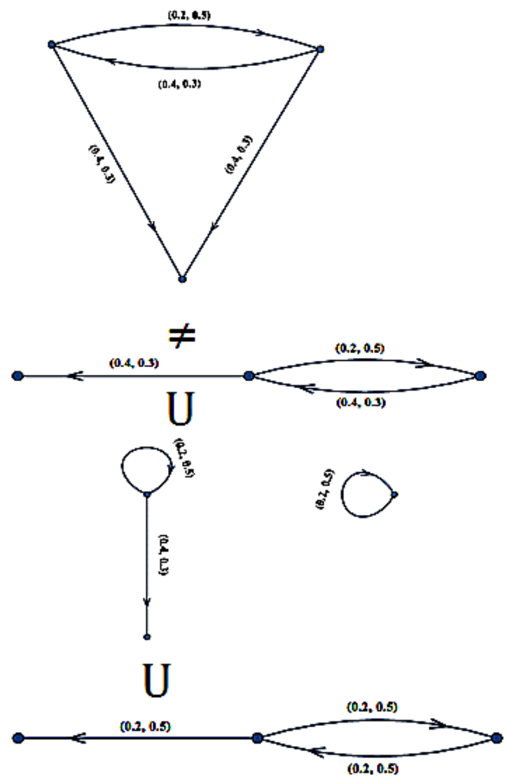

Figure 3: The graphical representation of equation (3)

$$S = Q \leq Q/Q.$$

(b) Let $n = 2$. Since S is nilpotent ($S^2 = \langle 0, 1\rangle$), we get

$$Q/Q \leq Q = S^+ = S \vee S^2 = S.$$

(c) Let $n \leq 3$. Suppose that $\langle s_{ij}, s'_{ij}\rangle < \langle t_{ij}, t'_{ij}\rangle$, then

$\langle t_{ij}, t'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$ and $\langle \bigvee\limits_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge\limits_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Now we get,

$\langle s_{il_1}, s'_{il_1}\rangle \wedge \langle s_{l_1l_2}, s'_{l_1l_2}\rangle \wedge ... \wedge \langle s_{l_hj}, s'_{l_hj}\rangle = \langle q_{ij}, q'_{ij}\rangle$ for suitable indices $l_1, l_2, ..., l_h (1 \leq h \leq n-2)$, so that,

$$\langle q_{il_1}, q'_{il_1}\rangle \wedge \langle q_{l_1l_2}, q'_{l_1l_2}\rangle \wedge ... \wedge \langle q_{l_hj}, q'_{l_hj}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Thus, $\langle q_{il_1}, q'_{il_1}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{l_1j}, q'_{l_1j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ for l_1. Then,

$$\langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

$$\left\langle t_{ij}, t'_{ij}\right\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (r_{ik} \wedge r_{kj}), \bigwedge_{k=1}^{n} (r'_{ik} \vee r'_{kj})\rangle = \langle 0, 1\rangle,$$

which contradicts the fact that $\langle t_{ij}, t'_{ij}\rangle > \langle 0, 1\rangle$. Thus, $T \leq S$, so that $Q/Q \leq S \leq Q$.

By the properties of Theorem 3.3 above, Q/Q is minimal in the set of IFMs such that

$$S^{+} = Q.$$

Theorem 3.4.

Let Q be an $n \times n$ irreflexive and transitive IFM. Then, the following conditions are equivalent:

(1) $Q/Q \leq S \leq Q$.

(2) $Q/Q = S/Q$.

Proof:

Let $F = (\langle f_{ij}, f'_{ij}\rangle) = Q/Q$ and $G = (\langle g_{ij}, g'_{ij}\rangle) = S/Q$. Then,

$$\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle,$$
$$\langle g_{ij}, g'_{ij}\rangle = \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge s_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee s'_{kj})\rangle.$$

(1) $\Rightarrow$ (2): Suppose that $Q/Q \leq S \leq Q$, so that $\langle f_{ij}, f'_{ij}\rangle \leq \langle s_{ij}, s'_{ij}\rangle \leq \langle r_{ij}, r'_{ij}\rangle$.

(a) First we show that, $\langle f_{ij}, f'_{ij}\rangle \leq \langle g_{ij}, g'_{ij}\rangle$. Let $\langle f_{ij}, f'_{ij}\rangle > \langle 0, 1\rangle$.

Then, $\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$, so that $\langle s_{ij}, s'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle$ and

$$\langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle.$$

Since $\langle q_{ik}, q'_{ik}\rangle \geq \langle s_{ik}, s'_{ik}\rangle$, we have $\langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Thus,

$$\begin{aligned}\langle g_{ij}, g'_{ij}\rangle &= \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kj})\rangle \\ &= \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kj})\rangle\end{aligned}$$

so that $\langle f_{ij}, f'_{ij}\rangle \leq \langle g_{ij}, g'_{ij}\rangle$.

(b) Next we show that $\langle g_{ij}, g'_{ij}\rangle \leq \langle f_{ij}, f'_{ij}\rangle$.

Let $\langle g_{ij}, g'_{ij}\rangle > \langle 0, 1\rangle$, then

$$\langle g_{ij}, g'_{ij}\rangle = s_{ij}, s'_{ij} > \langle 0,1\rangle, \text{ and hence,}$$

$$\langle \bigvee_{k=1}^{n}(s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik} \vee q'_{kj})\rangle < \langle q_{ij}, q'_{ij}\rangle.$$

Recall that,

$$\langle f_{ij}, f'_{ij}\rangle = \langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle \leq \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Since $\langle s_{ij}, s'_{ij}\rangle > \langle 0, 1\rangle$, we have $\langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle$.

We shall show that if $\langle f_{ij}, f'_{ij}\rangle < \langle q_{ij}, q'_{ij}\rangle$, then there is a contradiction.

Suppose that $\left\langle f_{ij}, f'_{ij}\right\rangle < \left\langle q_{ij}, q'_{ij}\right\rangle$. Then,

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle \geq \langle q_{ij}, q'_{ij}\rangle \geq \langle 0,1\rangle,$$

so that

$$\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \ \langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q^{(2)}_{ik(1)}, q'^{(2)}_{ik(1)}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle q_{ij}, q'_{ij}\rangle, \text{ for some k(1).}$$

We have $\langle s_{ik(1)}, s'_{ik(1)}\rangle \leq \langle q_{ij}, q'_{ij}\rangle$, since $\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kj})\rangle < \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Then, since $F \leq S$, we have $\langle f_{ik(1)}, f'_{ik(1)}\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Furthermore, $\langle f_{ik(1)}, f'_{ik(1)}\rangle < \langle q_{ik(1)}, q'_{ik(1)}\rangle$, since $\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

Thus,

$$\langle \bigvee_{k=1}^{n}(q_{ik} \wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(q'_{ik} \vee q'_{kk(1)})\rangle \geq \langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

Therefore,

$$\langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,\ \langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q_{ij}^{(3)}, q_{ij}'^{(3)}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle, \text{ for some k(2).}$$

Since $\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$ and $\langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$,

we have $\langle q_{k(2)j}, q'_{k(2)j}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, so that $\langle s_{ik(2)}, s'_{ik(2)}\rangle < \langle q_{ij}, q'_{ij}\rangle$, since

$$\langle \bigvee_{k=1}^{n} (s_{ik} \wedge r_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee r'_{kj})\rangle < \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle.$$

Then, since $F \leq S$, we have $\langle f_{ik(2)}, f'_{ik(2)}\rangle < \langle q_{ij}, q'_{ij}\rangle$.

Moreover, $\langle f_{ik(2)}, f'_{ik(2)}\rangle < \langle q_{ik(2)}, q'_{ik(2)}\rangle$, since $\langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$.

Thus,

$$\langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kk(2)}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kk(2)})\rangle \geq \langle q_{ik(2)}, q'_{ik(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle.$$

Therefore,

$$\langle q_{ik(3)}, q'_{ik(3)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle,$$

$$\langle q_{k(3)k(2)}, q'_{k(3)k(2)}\rangle \geq \langle q_{ij}, q'_{ij}\rangle, \text{ and}$$

$$\langle q_{ij}^{(4)}, q_{ij}'^{(4)}\rangle = \langle q_{ij}, q'_{ij}\rangle > \langle 0, 1\rangle, \text{ for a few } k(3).$$

By continuing the same process, we get $\langle q_{ij}^{(n)}, q_{ij}^{(n)}\rangle > \langle 0, 1\rangle$. That is a contradiction, since Q is nilpotent.

Thus, $\langle f_{ij}, f'_{ij}\rangle \geq \langle q_{ij}, q'_{ij}\rangle$, so that $\langle g_{ij}, g'_{ij}\rangle = \langle s_{ij}, s'_{ij}\rangle \leq \langle q_{ij}, q'_{ij}\rangle \leq \langle f_{ij}, f'_{ij}\rangle$.

(2) $\Rightarrow$ (1) Suppose that $Q/Q = S/Q$.

(a) It is evident that $Q/Q = S/Q \leq S$.

(b) We prove that $S \leq Q$. Suppose that $\langle s_{ij}, s'_{ij}\rangle > \langle q_{ij}, q'_{ij}\rangle$. Since

$$\langle q_{ij}, q'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kj})\rangle = \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kj}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kj})\rangle,$$

we have

$$\langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee s'_{kj})\rangle = \langle 0,1\rangle, \text{ so that}$$

$$\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kj}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kj})\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle.$$

Then,

$$\langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,$$

$$\langle q_{k(1)j}, q'_{k(1)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(1)}_{k(1)j}, q'^{(1)}_{k(1)j}\rangle = \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle, \text{ for a few } k(1).$$

Since

$$\langle q_{ij}, q'_{ij}\rangle < \langle s_{ij}, s'_{ij}\rangle \text{ and } \langle q_{ij}, q'_{ij}\rangle \geq \langle q_{ik(1)}, q'_{ik(1)}\rangle \wedge \langle q_{k(1)j}, q'_{k(1)j}\rangle,$$

we have $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ij}, s'_{ij}\rangle$, so that $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ik(1)}, s'_{ik(1)}\rangle$. Since

$$\begin{aligned}\langle q_{ik(1)}, q'_{ik(1)}\rangle &\leftharpoondown \langle \bigvee_{k=1}^{n}(q_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(q'_{ik}\vee q'_{kk(1)})\rangle\\ &= \langle s_{ij}, s'_{ij}\rangle \leftharpoondown \langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(1)})\rangle\end{aligned}$$

we get

$$\langle \bigvee_{k=1}^{n}(s_{ik}\wedge q_{kk(1)}), \bigwedge_{k=1}^{n}(s'_{ik}\vee q'_{kk(1)})\rangle \geq \langle s_{ik(1)}, s'_{ik(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle.$$

Then,

$$\langle s_{ik(2)}, s'_{ik(2)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle,$$

$$\langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle \geq \langle s_{ij}, s'_{ij}\rangle, \text{ and}$$

$$\langle q^{(2)}_{k(2)j}, q'^{(2)}_{k(2)j}\rangle \geq \langle s_{ij}, s'_{ij}\rangle > \langle 0,1\rangle, \text{ for a few } k(2).$$

Since $\langle q_{ik(1)}, q'_{ik(1)}\rangle < \langle s_{ij}, s'_{ij}\rangle$ and $\langle q_{ik(1)}, q'_{ik(1)}\rangle \geq \langle q_{ik(2)}, q'_{ik(2)}\rangle \wedge \langle q_{k(2)k(1)}, q'_{k(2)k(1)}\rangle$, we get

$$\langle q_{ik(2)}, q'_{ik(2)}\rangle < \langle s_{ij}, s'_{ij}\rangle, \text{ so that } \langle q_{ik(2)}, q'_{ik(2)}\rangle < \langle s_{ik(2)}, s'_{ik(2)}\rangle. \text{ Since}$$

$$\langle q_{ik(2)}, q'_{ik(2)} \rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (q_{ik} \wedge q_{kk(2)}), \bigwedge_{k=1}^{n} (q'_{ik} \vee q'_{kk(2)}) \rangle$$

$$= \langle s_{ik(2)}, s'_{ik(2)} \rangle \leftharpoondown \langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kk(2)}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kk(2)}) \rangle$$

we get $\langle \bigvee_{k=1}^{n} (s_{ik} \wedge q_{kk(2)}), \bigwedge_{k=1}^{n} (s'_{ik} \vee q'_{kk(2)}) \rangle \geq \langle s_{ik(2)}, s'_{ik(2)} \rangle \geq \langle s_{ij}, s'_{ij} \rangle$.

Then,

$$\langle s_{ik(3)}, s'_{ik(3)} \rangle \geq \langle s_{ij}, s'_{ij} \rangle,\ \langle q_{k(3)k(2)}, q'_{k(3)k(2)} \rangle \geq \langle s_{ij}, s'_{ij} \rangle, \text{ and}$$

$$\langle q^{(3)}_{k(3)j}, q'^{(3)}_{k(3)j} \rangle \geq \langle s_{ij}, s'_{ij} \rangle > \langle 0, 1 \rangle, \text{ for some } k(3).$$

By continuing the same argument,

$$\langle s_{ik(n)}, s'_{ik(n)} \rangle \geq \langle s_{ij}, s'_{ij} \rangle,\ \langle q_{k(n)k(n-1)}, q'_{k(n)k(n-1)} \rangle \geq \langle s_{ij}, s'_{ij} \rangle, \text{ and}$$

$$\langle q^{(n)}_{k(n)j}, q'^{(n)}_{k(n)j} \rangle \geq \langle s_{ij}, s'_{ij} \rangle > \langle 0, 1 \rangle,$$

which contradicts the fact Q is nilpotent. Hence $\langle s_{ij}, s'_{ij} \rangle < \langle q_{ij}, q'_{ij} \rangle$ for all i,j.

By Theorem 3.1, Theorem 3.3 and Theorem 3.4 we find the following Corollaries, which are handy for reduction of nilpotent IFMs or acyclic graphs. We should remember that if Q is nilpotent, then Q^+ is irreflexive and transitive.

Corollary 3.5.

If Q is nilpotent, then

(1) $Q^+/Q^+ \leq Q \leq Q^+$,

(2) $Q^+/Q^+ = Q/Q^+ = Q^+/Q$,

(3) $(Q^+/Q^+)^+ = Q^+$.

Proof:

(1) Is true by taking $Q = S$, by Theorem 3.3.

(2) By (1) and Theorem 3.4 we get

$$Q/Q^+ = Q^+/Q^+ = Q^+ \overset{c}{\leftharpoondown} (Q^2 \vee Q^3 \vee ... \vee Q^n - 1) = Q^+/Q.$$

(3) By Theorem 3.1.

Corollary 3.6.

Let Q be nilpotent. Then, the following statements are equivalent:

(1) $Q/Q^{+} \leq S \leq Q^{+}$,

(2) $Q^{+} = S^{+}$,

(3) $Q/Q^{+} = S/Q^{+}$.

4. Conclusion

In this article some properties of irreflexive, transitive intuitionistic fuzzy matrices are explored. Further, reduction of nilpotent IFM are obtained by applying implication operator. The concept of reduction of nilpotent IFMs are discussed.

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REFERENCES

Adak, A. K, Bhowmik, M and Pal, M. (2011). Application of generalized intuitionistic fuzzy matrix in multi-criteria decision making problem, Journal of Mathematical and Computational Science, Vol. 1, No. 1, pp. 19-31.

Adak, A. K, Bhowmik, M and Pal, M. (2012). Some properties of generalized intuitionistic fuzzy nilpotent matrices and its some properties, International Journal of Fuzzy Information and Engineering, Vol. 4, pp. 371-387.

Atanassov, K. (1987). Generalized index matrices, C. R. Acad Bulgare Sci., Vol. 40, No. 11, pp. 15-18.

Atanassov, K. (1994). Index matrix representation of the intuitionistic fuzzy garphs, Proceeding of Fifth Scientific Session of the Math Foundation of Artificial Intelligence Seminar, Sofia, 36-41. Accessed 5 Oct 1994 (Preprint MRL-MFAIS-10-94).

Atanassov, K. (1983). Intuitionistic fuzzy sets. VII ITKR's Session, Sofia, June.

Atanassov, K. (1999). Intuitionistic fuzzy sets; Theory and Applications. Physica Verlag.

Atanassov, K. (2005a). Intuitionistic fuzzy implications and modus ponens, Notes on Intuitionistic Fuzzy Sets, Vol. 11, No. 1, pp. 1-5.

Atanassov, K. (2005b). On some types of fuzzy negations, Notes on Intuitionistic Fuzzy Sets, Vol. 11, No. 4, pp. 170-172.

Atanassov, K. (2005c). A new intuitionistic fuzzy implication from a modal type, Advance Studies in Contemporary Mathematics, Vol. 12, No. 1, pp. 117-122.

Atanassov, K. and Gargov, G. (1998). Elements of intuitionistic fuzzy logic. Part I, Fuzzy Sets and Systems, Vol. 95, pp. 39-52.

Atanassov, K. (2014) Index matrices towards an augmented matrix calculus, Springer, Cham.

Bhowmik, M. and Pal, M. (2008a). Some results on intuitionistic fuzzy matrices and circulant intuitionistic fuzzy matrices, International Journal of Mathematical Sciences, Vol. 7, No. 1-2, pp. 81-96.

Bhowmik, M. and Pal, M. (2008b). Generalized intuitionistic fuzzy matrices, Far East Journal of Mathematical Sciences, Vol. 29, No. 3, pp. 533-554.

Han, S-C., Li, H-X. Wang, J-Y. (2005). On nilpotent incline matrices, Linear Algebra and its Applications, Vol. 406, pp. 201-217.

Hashimoto, H. (2005) Traces of fuzzy relations under dual operations, Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol. 9, No. 5, pp. 563-569.

Hashimoto, H. (1984). Sub-inverses of fuzzy matrices, Fuzzy Sets and Systems, Vol. 12, pp. 155-168.

Hashimoto, H. (1982a). Reduction of retrieval models, Information Science, Vol. 27, pp. 133-140.

Hashimoto, H. (1982b). Reduction of a nilpotent fuzzy matrix, Information Sciences, Vol. 27, pp. 233-243.

Im, Y, B. (2006). The determinant of square intuitionistic fuzzy matrices, Acta Ciencia Indica, XXXII M, pp. 515-524.

Lee. H. Y and Jeong. N. G. (2005). Canonical form of transitive intuitionistic fuzzy matrices, Honam Mathematical Journal, Vol. 27, No. 4), pp. 543-550.

Lur, Y. Y., Pang, C-T. and Guu, S-M. (2003). On simultaneously nilpotent fuzzy matrices, Linear Algebra and its Applications, Vol. 367, pp. 37-45.

Lur, Y. Y., Pang, C-T. and Guu, S-M. (2004). On nilpotent Fuzzy Matrices, Fuzzy Sets and System, Vol. 145, pp. 287-299.

Meenakshi, A. R. and Gandhimathi, T. (2010). Intuitionistic fuzzy relational equations, Advances in Fuzzy Mathematics, Vol. 5, No. 3, pp. 239-244.

Meenakshi, A. R. (2008). Fuzzy Matrix Theory And Applications. MJP Publishers, Chennai.

Mondal. S and Pal. M. (2013). Similarity relations, invertibility and eigenvalues of intuitionistic fuzzy matrix, International Journal of Fuzzy Information and Engineering, Vol. 4, pp. 431-443.

Mondal. S and Pal. M. (2014) Intuitionistic fuzzy incline matrix and determinant, Annals of Fuzzy Mathematics and Informatics, Vol. 8, No. 1, pp. 19-32.

Murugadas, P. and Lalitha, K. (2012). Dual implication operator in intuitionistic fuzzy matrices. Int. Conference on Mathematical Modeling and its Applications- Dec 22-24, Organized by Department of Mathematics, Annamalai University.

Murugadas, P. and Lalitha, K. (2014a). Bi-implication operator on intuitionistic fuzzy set, Journal of advances in Mathematics, Vol. 6, No. 2, pp. 961-969.

Murugadas, P. and Lalitha, K. (2014b). Sub-inverse and g-inverse of an intuitionistic fuzzy matrix using bi-implication operator, Int. Journal of Computer Application, Vol. 89, No. 1, pp. 1-5.

Murugadas, P. and Lalitha, K. (2014c). Implication operator on intuitionistic fuzzy tautological matrix, Int. Journal of Fuzzy Mathematical Archive, Vol. 5, No. 2, pp. 79-87.

Murugadas, P. and Lalitha, K. (2016). Decomposition of an intuitionistic fuzzy matrix using implication operators, Annals of Fuzzy Mathematics and Informatics, Vol. 11, No. 13, pp. 11-18.

Murugadas, P. (2011). Contribution to a study on generalized fuzzy matrices. Ph.D Thesis Department of Mathematics, Annamalai University.

Murugadas. P and Padder R. A. (2015). Reduction of an intuitionistic fuzzy rectangular Matrix, Annamalai University Science Journal, Vol. 49, pp. 15-18.

Ovchinnikov S. V. (1981), Structure of fuzzy binary relations, Fuzzy Sets and Systems, Vol. 6, pp. 169-195.

Padder R. A and Murugadas. P. (2016a). Max-max operation on intuitionistic fuzzy matrix, Annals of Fuzzy Mathematics and Informatics, Article in press.

Padder R. A and Murugadas. P. (2016b). Convergence of powers and canonical form of s-transitive intuitionistic fuzzy matrix, New Trends in Mathematical Sciences, Article in press.

Pal, M., Khan, S. k. and A. K Shyamal. (2002). Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, Vol. 8, No. 2, pp. 51-62.

Pradhan. R and Pal. M. (2013). Convergence of maxarithmetic mean-minarithmetic mean powers of intuitionistic fuzzy matrices, Intern. J. Fuzzy Mathematical Archive, Vol. 2, pp. 58-69.

Pradhan. R and Pal. M. (2012). Intuitionistic fuzzy linear transformations, Annals of Pure and Applied Mathematics, Vol. 1, No. 1, pp. 57-68.

Pradhan. R and Pal. M. (2014). The generalized inverse of Atanassov's intuitionistic fuzzy matrices, International Journal of Computational Intelligence Systems, Vol. 7, No. 6, pp. 1083-1095.

Shyamal, A.K. and Pal. M. (2002). Distances between intuitionistic fuzzy matrices, V.U.J. Physical Sciences, Vol. 8, pp. 81-91.

Sriram, S. and Murugadas, P. (2011). Sub-inverses of intuitionistic fuzzy matrices, Acta Ciencia Indica (Mathematics), Vol. XXXVII, M N0 1, pp. 41-56.

Sriram, S. and Murugadas, P. (2010). On semi-ring of intuitionistic fuzzy matrices, Applied Mathematical Science, Vol. 4, No. 23, pp. 1099-1105.

Tan, Y-J. (2005). On nilpotent matrices over distributive Lattices, Fuzzy Sets and System, Vol. 151, pp. 421-433.

Xu, Z. and Yager, R.R. (2006). Some geometric operators based on Intuitionistic Fuzzy Sets, Int. Journal of General Systems, Vol. 35, pp. 417-433.

Zadeh. L.A. (1965). Fuzzy Sets, Journal of Information and Control, Vol. 8, pp. 338-353.

Zadeh. L.A. (1971). Similarity relations and fuzzy orderings, Inform. Sci., Vol. 3, pp. 177-200.