Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466
Vol. 11, Issue 2 (December 2016), pp. 599 – 613

Applications and Applied Mathematics:
An International Journal
(AAM)

Random variational-like inclusion and random proximal operator equation for random fuzzy mappings in Banach spaces

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Received: May 4, 2016; Accepted: August 8, 2016

Abstract

In this paper, we introduce and study a random variational-like inclusion and its corresponding random proximal operator equation for random fuzzy mappings. It is established that the random variational-like inclusion problem for random fuzzy mappings is equivalent to a random fixed point problem. We also establish a relationship between random variational-like inclusion and random proximal operator equation for random fuzzy mappings. This equivalence is used to define an iterative algorithm for solving random proximal operator equation for random fuzzy mappings. Through an example, we show that the random Wardrop equilibrium problem is a special case of the random variational-like inclusion problem for random fuzzy mappings.

Keywords: Algorithm; Convergence; Inclusion; Proximal; Random

MSC 2010 No.: 47S40, 47H06

1. Introduction

Variational inclusions and variational-like inclusions are mathematical models of many problems arising in mechanics, physics, structural analysis, optimization, economics and engineering sci-

ences; for example, see Ahmad et al. (2005), Agarwal et al. (2000), Agarwal et al. (2002), Ding et al. (2000) and references therein. An important area of mathematics, fuzzy set theory, which was introduced by Zadeh (1965), is quite application-oriented and applied in many branches of mathematical and engineering sciences including artificial intelligence, management sciences, computer sciences (see e.g., Zimmermann (1988)). For more details we refer to Deepmala (2014), Farajzadeh et al. (2016), Husain et al. (2013), Mishra (2007), Husain et al. (2013), and Murthy et al. (2016).

Ansari (1988) as well as Chang et al. (1989) first introduced the concept of variational inequalities for fuzzy mappings. Since then several types of variational inequalities with fuzzy mappings were considered by Noor (1989), Chang et al. (1993), Ding et al. (2002) and Park et al. (2000) in Hilbert spaces. Further, Huang et al. (2005) considered nonlinear equations in fuzzy normed spaces and Lan et al. (2008) considered fuzzy variational inclusions in Banach spaces, and Ahmad et al. (2015) studied generalized variational inclusion with fuzzy mappings.

The uncertain events occurring in decision-making problems are characterized by fuzzy uncertainty as well as randomness. Fuzzy random uncertainty appears in many real world applications such as inventory management, facilities planning, transportation assignment, equilibrium problem for economics, and migration theory.

The concept of random variational inequality is the cardinal part of random functional analysis. These topics have attracted many researchers due to the major applications of random problems (e.g. see Jin et al. (2004), Lan et al. (2004), Lan et al. (2004), Lan et al. (2003), Liu et al. (2001), Lan (2006), Verma et al. (2005)). The random variational inequality and random quasi variational inequality problems have been introduced and studied by Chang et al. (1993), Cho et al. (2000), and Huang et al. (1999).

After that Ahmad et al. (2005) and Ahmad et al. (2009) studied a class of random variational inclusions with random fuzzy mappings in different settings.

Recently, Lan et al. (2009) studied a class of general nonlinear random multivalued operator equations involving generalized m-accretive mappings in Banach spaces and Onjai-Uea et al. (2010) studied nonlinear random equations with random fuzzy mappings in uniformly smooth Banach spaces.

Up to the best of our knowledge, nobody has yet studied random proximal operator equation for random fuzzy mappings. That is the main objective of this paper. We define an iterative algorithm for finding the random solution of the random proximal operator equation for random fuzzy mappings which is also the random solution of random variational-like inclusion problem.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space with its norm $\|.\|$, E^* is the topological dual of E, and d is the metric induced by the norm $\|.\|$. Let $\mathcal{F}(E)$ be the collection of all fuzzy sets over E. A mapping F from E to $\mathcal{F}(E)$ is called a fuzzy mapping on E. If F

is a fuzzy mapping on E, then $F(x)$ (we denote it by F_x in the sequel) is a fuzzy set on E and $F_x(y)$ is the membership function of y in F_x. Let $N \in \mathcal{F}(E)$, $q \in [0,1]$. Then the set

$$(N)_q = \{x \in E : N(x) \geq q\}$$

is called a q-cut set of N.

We denote by (Ω, Σ) a measurable space, where Ω is a set and Σ is a σ-algebra of subsets of Ω and by $\mathcal{B}(E)$, 2^E, $CB(E)$ and $D(.,.)$, the class of Borel σ-fields in E, the family of all nonempty subsets of E, the family of all nonempty closed bounded subsets of E, and the Hausdörff metric on $CB(E)$, respectively.

Definition.

A mapping $x : \Omega \to E$ is said to be measurable if for any $B \in \mathcal{B}(E)$, $\{t \in \Omega : x(t) \in B\} \in \Sigma$.

Definition.

A mapping $f : \Omega \times E \to E$ is called a random operator if for any $x \in E$, $f(t,x) = x(t)$ is measurable. A random operator f is said to be continuous if for any $t \in \Omega$, the mapping $f(t,.) : E \to E$ is continuous.

Definition.

A multi-valued mapping $T : \Omega \to 2^E$ is said to be measurable if for any $B \in \mathcal{B}(E)$, $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \varnothing\} \in \Sigma$.

Definition.

A mapping $u : \Omega \to E$ is called a measurable selection of a multi-valued measurable mapping $T : \Omega \to 2^E$ if u is measurable and for any $t \in \Omega$, $u(t) \in T(t)$.

Definition.

A mapping $T : \Omega \times E \to 2^E$ is called a random multi-valued mapping if for any $x \in E$, $T(.,x)$ is measurable. A random multi-valued mapping $T : \Omega \times E \to CB(E)$ is said to be D-continuous if for any $t \in \Omega$, $T(t,.)$ is continuous in the Hausdörff metric.

Definition.

A fuzzy mapping $F : \Omega \to \mathcal{F}(E)$ is called measurable, if for any $\alpha \in (0,1]$, $(F(.))_\alpha : \Omega \to 2^E$ is a measurable multi-valued mapping.

Definition.

A fuzzy mapping $F : \Omega \times E \to \mathcal{F}(E)$ is called a random fuzzy mapping, if for any $x \in E$, $F(.,x) : \Omega \to \mathcal{F}(E)$ is a measurable fuzzy mapping.

It is to be noted that multi-valued mappings, random multi-valued mappings and fuzzy mappings are special cases of random fuzzy mappings.

We assume that $\langle \cdot, \cdot \rangle$ is the duality pairing between E and E^* and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\| \text{ and } \|f\| = \|x\|\}, \text{ for all } x \in E.$$

Proposition.

Let E be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \ \forall j(x+y) \in J(x+y).$$

Let $M, S, T : \Omega \times E \to \mathcal{F}(E)$ be three random fuzzy mappings satisfying the following condition:

(A) There exists three mappings $a, b, c : E \to [0,1]$ such that

$$(M_{t,x})_{a(x)} \in CB(E), \ (S_{t,x})_{b(x)} \in CB(E), \ (T_{t,x})_{c(x)} \in CB(E), \ \forall \ (t,x) \in \Omega \times E.$$

By using the random fuzzy mappings M, S and T, we can define three random multi-valued mappings $\tilde{M}$, $\tilde{S}$, and $\tilde{T}$ as

$$\tilde{M} : \Omega \times E \to CB(E), \ x \to (M_{t,x})_{a(x)}, \ \forall \ (t,x) \in \Omega \times E,$$

$$\tilde{S} : \Omega \times E \to CB(E), \ x \to (S_{t,x})_{b(x)}, \ \forall \ (t,x) \in \Omega \times E,$$

and

$$\tilde{T} : \Omega \times E \to CB(E), \ x \to (T_{t,x})_{c(x)}, \ \forall \ (t,x) \in \Omega \times E,$$

where $N_{t,x} = N(t, x(t))$.

In the sequel $\tilde{M}$, $\tilde{S}$, and $\tilde{T}$ are called the random multi-valued mappings induced by the random fuzzy mappings M, S and T, respectively.

Given mappings $a, b, c : E \to [0,1]$, random fuzzy mappings $M, S, T : \Omega \times E \to \mathcal{F}(E)$, random operators $N : \Omega \times E \times E \to E^*$, $\eta : \Omega \times E \times E \to E$, $g : \Omega \times E \to E$ and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, η-subdifferentiable lower continuous functional (may not be convex) such that $Im(g) \cap dom(\partial_\eta \phi) \neq \varnothing$, where $\partial_\eta \phi$ denotes the η-subdifferential of ϕ. We consider the following problem:

Find measurable mappings $x, u, v, w : \Omega \to E$ such that for all $t \in \Omega$, $x(t) \in E$, $y(t) \in E$, $M_{t,x(t)}(u(t)) \geq a(x(t))$, $S_{t,x(t)}(v(t)) \geq b(x(t))$, $T_{t,x(t)}(w(t)) \geq c(x(t))$, $g(t, w(t)) \cap dom(\partial_\eta \phi) \neq \varnothing$, and

$$\langle N(t, u(t), v(t)), \eta(t, y(t), g(t, w(t)))\rangle \geq \phi(g(t, w(t))) - \phi(y(t)). \tag{1}$$

Problem (1) is called the random variational-like inclusion problem for random fuzzy mappings. The set of measurable mappings (x, u, v, w) is called a random solution of (1).

If E a Hilbert space and $N(t, u(t), v(t)) = P(t, u(t)) - \{f(t, v(t)) - g(t, w(t))\}$ and $\eta(t, y(t), w(t)) = y(t) - g(t, w(t))$, where $P, f, g : \Omega \times E \to E$ be the random operators, then problem (1) reduces to the problem (1) of Ahmad et al. (2005).

We remark that random variational-like inclusion problem for random fuzzy mappings (1) includes as special cases, many kinds of variational inequalities and quasi-variational inequalities for random fuzzy mappings.

Through the following example, we show that the random Wardrop equilibrium problem can be obtained from random variational-like inclusion problem for random fuzzy mappings (1).

Example 2.1.

In a traffic network model, a path flow H is called an equilibrium flow or Wardrop equilibrium if and only if $H \in K$ and for any paths R_q, R_s there holds

$$C_q(H) < C_s(H) \Rightarrow H_s = 0,$$

which is equivalent to the following problem:

$$\textit{Find} \;\; H \in K \textit{ such that } \langle C(H), F - H \rangle \geq 0, \forall \; F \in K, \tag{2}$$

where

$$K = \{F \in \mathbb{R}^k : F_r \geq 0, \textit{ for any } r = 1, 2, \cdots, k \textit{ and } \phi F = D\},$$

ϕ is the pair-path incidence matrix and D is the traffic demand vector, F is path network flow, and C is the cost on the paths. The random version of (2) was introduced and studied by Gwinner et al. (2010), which results from an uncertain demand and uncertain costs. For the path-flow variables, the random Wardrop equilibrium problem is to find $H(t) \in K(t)$ such that

$$\langle C(t, H(t)), F - H(t) \rangle \geq 0, \forall F \in K(t), \tag{3}$$

where for any $t \in \Omega$,

$$K(t) = \{F \in \mathbb{R}^k : F_r \geq 0, \text{ for any } r = 1, 2, \cdots, k \text{ and } \phi(F) = D(t)\}.$$

For E to be a Hilbert space, $N(t, u(t), v(t)) = C(t, H(t))$, $\eta(t, y(t), g(t, w(t))) = F - H(t)$ and $\phi = 0$. Then random variational-like inclusion problem with random fuzzy mapping (1) reduces to random Wardrop equilibrium problem (3).

Definition.

A random operator $g : \Omega \times E \to E$ is said to be Lipschitz continuous, if there exists a measurable function $\lambda_g : \Omega \to (0, \infty)$ such that

$$\|g(t, x_1(t)) - g(t, x_2(t))\| \leq \lambda_{g(t)} \|x_1(t) - x_2(t)\|, \; \forall x_1(t), x_2(t) \in E, t \in \Omega.$$

Definition.

A random multi-valued mapping $T : \Omega \times E \to CB(E)$ is said to be

(i) Strongly accretive with respect to a random operator $g : \Omega \times E \to E$, if there exists a measurable function $k : \Omega \to (0, \infty)$ such that

$$\begin{aligned} &\langle g(t, w_1(t)) - g(t, w_2(t)),\ j(x_1(t) - x_2(t))\rangle \geq k(t)\|x_1(t) - x_2(t)\|^2, \\ &\forall\ x_1(t), x_2(t) \in E, w_1(t) \in T(t, x_1(t)), w_2(t) \in T(t, x_2(t)), \\ &j(x_1(t) - x_2(t)) \in J(x_1(t) - x_2(t))\ and\ t \in \Omega; \end{aligned}$$

(ii) D-Lipschitz continuous, if there exists a measurable function $\delta_T : \Omega \to (0, \infty)$ such that

$$D(T(t, x_1(t)), T(t, x_2(t))) \leq \delta_T\|x_1(t) - x_2(t)\|, \forall\ x_1(t), x_2(t) \in E.$$

Definition.

Let $J : \Omega \times E \to E^*$, $\eta : \Omega \times E \times E \to E$ be the random mappings. Then

(i) J is said to be η-strongly accretive, if there exists a measurable function $\alpha : \Omega \to (0, \infty)$ such that

$$\langle J_t(x_1(t)) - J_t(x_2(t)), \eta(t, x_1(t), x_2(t))\rangle \geq \alpha(t)\|x_1(t) - x_2(t)\|^2, \forall\ x_1(t), x_2(t) \in E;$$

(ii) η is said to be Lipschitz continuous, if there exists a measurable function $\tau : \Omega \to (0, \infty)$ such that

$$\|\eta(t, x_1(t), x_2(t)) \leq \tau(t)\|x_1(t) - x_2(t)\|, \forall\ x_1(t), x_2(t) \in E.$$

Definition.

A functional $f : E \times E \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short 0-DQCV) in y, if for any finite subset $\{x_1, x_2, \cdots, x_n\} \subset E$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0.$$

Definition.

Let $N : \Omega \times E \times E \to E^*$ be a random operator and $\tilde{M} : \Omega \times E \to CB(E)$ be a random multi-valued mapping. Then N is said to be Lipschitz continuous with respect to $\tilde{M}$ if there exists a measurable function $\lambda_{N_{\tilde{M}}} : \Omega \to (0, \infty)$ such that

$$\begin{aligned} \|N(t, u_t(t), .) - N(t, u_{n-1}(t), .)\| \ \leq\ & \lambda_{N_{\tilde{M}}(t)}\|u_n(t) - u_{n-1}(t)\|, \forall\ u_n(t) \in \tilde{M}(t, x_n(t)), \\ & u_{n-1} \in \tilde{M}(t, x_{n-1}(t)). \end{aligned}$$

3. Random Iterative Algorithm and Convergence Theorem

We begin this section with the following Lemmas.

Lemma 1. [Ahmad et al. (2005)]

Let $T : \Omega \times E \to CB(E)$ be a D-continuous random multi-valued mapping. Then for any measurable mapping $w : \Omega \to E$, the multi-valued mapping $T(., w(.)) : \Omega \to CB(E)$ is measurable.

Lemma 2. [Ahmad et al. (2005)]

Let $S, T : \Omega \to CB(E)$ be two measurable multi-valued mappings, $\epsilon > 0$ be a constant and $v : \Omega \to E$, be a measurable selection of S. Then there exist a measurable selection $w : \Omega \to E$ of T such that for all $t \in \Omega$,

$$\|v(t) - w(t)\| \leq (1 + \epsilon)D(S(t), T(t)).$$

We have the following definition of random proximal operator.

Definition.

Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous (may not be convex) and η-subdifferentiable functional. Let $\eta : \Omega \times E \times E \to E$ and $J : \Omega \times E \to E^*$ be the mappings. Then the random proximal operator denoted by $J^{\partial_\eta \phi}_{\rho(t)}$ is defined by

$$J^{\partial_\eta \phi}_{\rho(t)}(x^*(t)) = [J_t + \rho(t)\partial_\eta \phi]^{-1}(x^*(t)), \ \forall \ x^*(t) \in E^*. \tag{4}$$

Theorem 1.

Let E be a reflexive Banach space with its dual space E^* and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable, proper functional which may not be convex. Let $J : \Omega \times E \to E^*$ be η-strongly accretive with constant $\alpha(t) > 0$. Let $\eta : \Omega \times E \times E \to E$ be Lipschitz continuous with constant $\tau(t) > 0$ such that $\eta(t, x(t), y(t)) = -\eta(t, y(t), x(t))$, for all $t \in \Omega$, $x(t), y(t) \in E$ and for any $x(t) \in E$, the function $h(t, y(t), x(t)) = \langle x^*(t) - J_t(x(t)), \eta(t, y(t), x(t))\rangle$ is 0-DQCV in $y(t)$. Then for any $\rho(t) > 0$, and for any $x^*(t) \in E^*$, there exists a unique $x \in E$ such that

$$\langle J_t(x(t)) - x^*(t), \eta(t, y(t), x(t))\rangle + \rho\phi(y(t)) - \rho(t)\phi(x(t)) \geq 0, \ for \ all \ y(t) \in E,$$

where $\alpha, \tau, \rho : \Omega \to (0, \infty)$ are the measurable functions.

That is, $x(t) = J^{\partial_\eta \phi}_{\rho(t)}(x^*(t))$ and so that random proximal mapping of ϕ is well defined and $\frac{\tau(t)}{\alpha(t)}$-Lipschitz continuous.

Proof:

For the proof, we refer to Ahmad et al. (2005).□

Lemma 3.

The set of measurable mappings $x, u, v, w : \Omega \to E$ is a random solution of problem (1) if and only if for all $t \in \Omega$, $x(t) \in E$, $u(t) \in \tilde{M}(t, x(t))$, $v(t) \in \tilde{S}(t, x(t))$, $w(t) \in \tilde{T}(t, x(t))$ and

$$g(t, w(t)) = J^{\partial_\eta \phi}_{\rho(t)}[J_t(g(t, w(t))) - \rho(t)N(t, u(t), v(t))],$$

where $\rho : \Omega \to (0, \infty)$ is a measurable function.

Proof:

The proof directly follows from the definition of $J^{\partial_\eta \phi}_{\rho(t)}$. $\square$

In connection with problem (1), we consider the following random proximal operator equation for random fuzzy mappings.

Find $z(t) \in E^*$, $x(t) \in E$, $u(t) \in \tilde{M}(t, x(t))$, $v(t) \in \tilde{S}(t, x(t))$ and $w(t) \in \tilde{T}(t, x(t))$ such that

$$N(t, u(t), v(t)) + \rho^{-1}(t) R^{\partial_\eta \phi}_{\rho(t)}(z(t)) = 0, \tag{5}$$

where $\rho(t)$ is same as in Theorem 1, $R^{\partial_\eta \phi}_{\rho(t)}(z(t)) = \left[I - J_t[J^{\partial_\eta \phi}_{\rho(t)}]\right](z(t))$, $J_t[J^{\partial_\eta \phi}_{\rho(t)}(z(t))] = [J_t(J^{\partial_\eta \phi}_{\rho(t)})](z(t))$ and I is the identity mapping.

Lemma 4.

The random variational-like inclusion problem (1) for random fuzzy mappings has a solution $(x(t), u(t), v(t), w(t))$ with $x(t) \in E$, $u(t) \in \tilde{M}(t, x(t))$, $v(t) \in \tilde{S}(t, x(t))$ and $w(t) \in \tilde{T}(t, x(t))$ if and only if random proximal operator equation for random fuzzy mappings (5) has a solution $(z(t), x(t), u(t), v(t), w(t))$ with $z(t) \in E^*$, $x(t) \in E$, $u(t) \in \tilde{M}(t, x(t))$, $v(t) \in \tilde{S}(t, x(t))$ and $w(t) \in \tilde{T}(t, x(t))$, where

$$g(t, w(t)) = J^{\partial_\eta \phi}_{\rho(t)}(z(t)),$$

$z(t) = J_t(g(t, w(t))) - \rho(t)[N(t, u(t), v(t))]$ and the operator J is one-one.

Proof:

Let $(x(t), u(t), v(t), w(t))$ be a random solution of problem (1). Then by Lemma 3, it is a solution of the following equation:

$$g(t, w(t)) = J^{\partial_\eta \phi}_{\rho(t)}[J_t(g(t, w(t))) - \rho(t)N(t, u(t), v(t))]. \tag{6}$$

Using the fact that $R^{\partial_\eta \phi}_{\rho(t)} = [I - J_t(J^{\partial_\eta \phi}_{\rho(t)})]$ and the above equation, we have

$$\begin{aligned}
&R^{\partial_\eta\phi}_{\rho(t)}[J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t))] \\
&\quad = J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t)) - J_t[J^{\partial_\eta\phi}_{\rho(t)}(J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t)))] \\
&\quad = J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t)) - J_t(g(t,w(t))) \\
&\quad = -\rho(t)N(t,u(t),v(t)),
\end{aligned}$$

which implies that

$$N(t,u(t),v(t)) + \rho^{-1}(t)R^{\partial_\eta\phi}_{\rho(t)}(z(t)) = 0,$$

i.e., $(x(t),u(t),v(t),w(t))$ is a random solution of problem (5).

Conversely, let $(z(t),x(t),u(t),v(t),w(t))$ be a random solution of problem (5), then we have

$$\rho(t)[N(t,u(t),v(t))] = -R^{\partial_\eta\phi}_{\rho(t)}(z(t)) = J_t[R^{\partial_\eta\phi}_{\rho(t)}(z(t))] - z(t). \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}
\rho(t)[N(t,u(t),v(t))] &= J_t\big[J^{\partial_\eta\phi}_{\rho(t)}\big(J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t))\big)\big] \\
&\quad -J_t(g(t,w(t))) + \rho(t)N(t,u(t),v(t)) \\
J_t(g(t,w(t))) &= J_t\big[J^{\partial_\eta\phi}_{\rho(t)}\big(J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t))\big)\big],
\end{aligned}$$

since J is one-one, we have

$$g(t,w(t)) = J^{\partial_\eta\phi}_{\rho(t)}[J_t(g(t,w(t))) - \rho(t)N(t,u(t),v(t))],$$

i.e., $(x(t),u(t),v(t),w(t))$ is a random solution of problem (1). $\square$

Using Lemmas 3 and 4, we define the following random iterative algorithm for solving random proximal operator equation for random fuzzy mappings (5).

Random Iterative Algorithm 3.1.

Suppose that $M,S,T:\Omega\times E\to\mathcal{F}(E)$ be three random fuzzy mappings satisfying condition (A). Let $\tilde{M},\tilde{S},\tilde{T}:\Omega\times E\to CB(E)$ be D-continuous random multi-valued mappings induced by M,S and T, respectively. Let $N:\Omega\times E\times E\to E^*$, $\eta:\Omega\times E\times E\to E$ and $g:\Omega\times E\to E$ be the continuous random mappings and $\phi:E\to\mathbb{R}\cup\{+\infty\}$ is a proper, η-subdifferentiable, lower semicontinuous (may not be convex) functional with $Im(g)\cap(dom\partial_\eta\phi)\neq\varnothing$, where $\partial_\eta\phi$ is the η-subdifferential of ϕ. For any given measurable mapping $x_0:\Omega\to E$, the multi-valued mappings $\tilde{M}(.,x_0(.)),\tilde{S}(.,x_0(.))$ and $\tilde{T}(.,x_0(.))$ are measurable by Lemma 1. Hence there exist measurable selection $u_0:\Omega\to E$ of $\tilde{M}(.,x_0(.))$, $v_0:\Omega\to E$ of $\tilde{S}(.,x_0(.))$ and $w_0:\Omega\to E$ of $\tilde{T}(.,x_0(.))$ by Himmelberg (1975). Let

$$z_1(t) = J_t(g(t,w_0(t))) - \rho(t)N(t,u_0(t),v_0(t)) \in E^* \ and \ x_1(t)\in E.$$

$$g(t, w_1(t)) = J^{\partial_\eta \phi}_{\rho(t)}(z_1(t)).$$

It is easy to see that $x_1 : \Omega \to E$ is measurable. By Lemma 3, there exist measurable selection $u_1 : \Omega \to E$ of $\tilde{M}(., x_1(t))$, $v_1 : \Omega \to E$ of $\tilde{S}(., x_1(t))$ and $w_1 : \Omega \to E$ of $\tilde{T}(., x_1(t))$ such that for all $t \in \Omega$

$$\begin{aligned} g(t, w_1(t)) &= J^{\partial_\eta \phi}_{\rho(t)}(z_1(t)) \\ \|u_0(t) - u_1(t)\| &\leq D(\tilde{M}(t, x_0(t)), \tilde{M}(t, x_1(t))), \\ \|v_0(t) - v_1(t)\| &\leq D(\tilde{S}(t, x_0(t)), \tilde{S}(t, x_1(t))), \\ \textit{and} \qquad \|w_0(t) - w_1(t)\| &\leq D(\tilde{T}(t, x_0(t)), \tilde{T}(t, x_1(t))). \end{aligned}$$

Let $z_2(t) = J_t(g(t, w_1(t))) - \rho(t)N(t, u_1(t), v_1(t)) \in E^*$ and $g(t, w_2(t)) = J^{\partial_\eta \phi}_{\rho(t)}(z_2(t))$.

Continuing the above process inductively, we obtain the following scheme:

For any $z_0(t) \in E^*$, $u_0(t) \in \tilde{M}(t, x_0(t))$, $v_0(t) \in \tilde{S}(t, x_0(t))$ and $w_0(t) \in \tilde{T}(t, x_0(t))$,

compute the random iterative sequences for solving problem (5) as:

$$(i) \quad g(t, w_n(t)) = J^{\partial_\eta \phi}_{\rho(t)}(z_n(t)), \tag{8}$$

$$(ii) \quad u_n(t) \in \tilde{M}(t, x_n(t)), \|u_n(t) - u_{n+1}(t)\| \leq D\big(\tilde{M}(t, x_n(t)), \tilde{M}(t, x_{n+1}(t))\big), \tag{9}$$

$$(iii) \quad v_n(t) \in \tilde{S}(t, x_n(t)), \quad \|v_t(t) - v_{n+1}(t)\| \leq D\big(\tilde{S}(t, x_n(t)), \tilde{S}(t, x_{n+1}(t))\big), \tag{10}$$

$$(iv) \quad w_n(t) \in \tilde{T}(t, x_n(t)), \|w_t(t) - w_{n+1}(t)\| \leq D\big(\tilde{T}(t, x_n(t)), \tilde{T}(t, x_{n+1}(t))\big), \tag{11}$$

$$(v) \quad z_{n+1}(t) = J_t(g(t, w_n(t))) - \rho(t)N(t, u_n(t), u_n(t), v_n(t)), \ n = 0, 1, 2 \cdots \ \textit{and } t \in \Omega. \tag{12}$$

In the following, we are now able to establish the convergence of random iterative sequences generated by the Random Iterative Algorithm 3.1.

Theorem 2.

Let E be a reflexive Banach space with the dual space E^*. Let the random operator $J : \Omega \times E \to E^*$ be η-strongly accretive with the constant $\alpha(t) > 0$ and Lipschitz continuous with constant $\lambda_{J(t)} > 0$. Let the random mapping $\eta : \Omega \times E \times E \to E$ be Lipschitz continuous with constant $\tau(t) > 0$ such that $\eta(t, x(t), y(t)) = -\eta(t, y(t), x(t))$, for all $t \in \Omega$, $x(t), y(t) \in E$ and $g : \Omega \times E \to E$ be Lipschitz continuous with constant $\lambda_{g(t)} > 0$. Suppose that $M, S, T : \Omega \times E \to \mathcal{F}(E)$ be the random fuzzy mappings satisfying the condition (A). Let $\tilde{M}, \tilde{S}, \tilde{T} : \Omega \times E \to CB(E)$ be the random multi-valued mappings induced by M, S and T, respectively such that $\tilde{M}$, $\tilde{S}$ and $\tilde{T}$ are D-Lipschitz continuous with constants $\delta_{\tilde{M}(t)}$, $\delta_{\tilde{S}(t)}$ and $\delta_{\tilde{T}(t)}$, respectively, $\tilde{T}$ is $k(t)$-strongly accretive with respect to g. Let the random mapping $N : \Omega \times E \times E \to E^*$ is Lipschitz continuous with respect to $\tilde{M}$ and $\tilde{S}$ with constants $\lambda_{N_{\tilde{M}(t)}}$ and $\lambda_{N_{\tilde{S}(t)}}$, respectively. Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower-semicontinuous, η-subdifferentiable, proper functional which may not be convex such that $g(t, w(t)) \cap dom(\partial_\eta \phi) \neq \varnothing$. Let for any $x(t) \in E$, the function $h(t, y(t), x(t)) = \langle x^*(t) - J_t(x(t)), \eta(t, y(t), x(t))\rangle$ is 0-DQCV in $y(t)$. If for any $\rho(t) > 0$, if

the following condition holds:

$$0 < [\lambda_{J(t)}\lambda_{g(t)}\lambda_{T(t)} + \lambda_{N\tilde{M}(t)}\delta_{\tilde{M}(t)} + \lambda_{N\tilde{S}(t)}\delta_{\tilde{S}(t)}] < \frac{\alpha(t)\sqrt{1+2k(t)}}{\tau(t)}, \tag{13}$$

then there exist $z(t) \in E^*$, $x(t) \in E$, $u(t) \in \tilde{M}(t, x(t))$, $v(t) \in \tilde{S}(t, x(t))$ and $w(t) \in \tilde{T}(t, x(t))$ satisfying random proximal operator equation (5) such that $z_n(t) \to z(t)$, $x_n(t) \to x(t)$, $u_n(t) \to u(t)$, $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$, where $\{z_n(t)\}$, $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are the random sequences obtained by Algorithm 3.1.

Proof:

From Algorithm 3.1, we have

$$\begin{aligned} \|z_{n+1}(t) - z_n(t)\| &= \|J_t(g(t, w_n(t))) - \rho N(t, u_n(t), v_n(t)) - \\ &\quad [J_t(g(t, w_{n-1}(t))) - \rho(t)N(t, u_{n-1}(t), v_{n-1}(t))]\| \\ &\leq \|J_t(g(t, w_n(t))) - J_t(g(t, w_{n-1}(t)))\| + \\ &\quad \rho(t)\|N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\|. \end{aligned} \tag{14}$$

By the Lipschitz continuity of J, g, D-Lipschitz continuity of $\tilde{T}$ and (3.7), we have

$$\begin{aligned} \|J_t(g(t, w_n(t))) - J_t(g(t, w_{n-1}(t)))\| &\leq \lambda_{J(t)}\|g(t, w_n(t)) - g(t, w_{n-1}(t))\| \\ &\leq \lambda_{J(t)}\lambda_{g(t)}\|w_n(t) - w_{n-1}(t)\| \\ &\leq \lambda_{J(t)}\lambda_{g(t)}D(\tilde{T}(t, x_n(t)), \tilde{T}(t, x_{n-1}(t))) \\ &\leq \lambda_{J(t)}\lambda_{g(t)}\delta_{\tilde{T}(t)}\|x_n(t) - x_{n-1}(t)\|. \end{aligned} \tag{15}$$

Since N is Lipschitz continuous with respect to $\tilde{M}$ and $\tilde{S}$ and D-Lipschitz continuous of $\tilde{M}$ and $\tilde{S}$, we have

$$\begin{aligned} \|N(t, u_n(t), v_n(t)) &- N(t, u_{n-1}(t), v_{n-1}(t))\| \\ &= \|N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_n(t)) \\ &\quad + N(t, u_{n-1}(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\| \\ &\leq \|N(t, u_n(t), v_n(t)) - N(t, u_{n-1}(t), v_n(t))\| \\ &\quad + \|N(t, u_{n-1}(t), v_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t))\| \\ &\leq \lambda_{N_{\tilde{M}(t)}}\|u_n(t) - u_{n-1}(t)\| + \lambda_{N_{\tilde{S}(t)}}\|v_n(t) - v_{n-1}(t)\| \\ &\leq \lambda_{N_{\tilde{M}(t)}}D(\tilde{M}(t, x_n(t)), \tilde{M}(t, x_{n-1}(t))) \\ &\quad + \lambda_{N_{\tilde{S}(t)}}D(\tilde{S}(t, x_n(t)), \tilde{S}(t, x_{n-1}(t))) \\ &\leq \left(\lambda_{N_{\tilde{M}(t)}}\delta_{\tilde{M}(t)} + \lambda_{N_{\tilde{S}(t)}}\delta_{\tilde{S}(t)}\right)\|x_n(t) - x_{n-1}(t)\|. \end{aligned} \tag{16}$$

Combining (15) and (16) with (14), we have

$$\|z_{n+1}(t) - z_n(t)\| \leq \left[\lambda_{J(t)}\lambda_{g(t)}\delta_{T(t)} + \lambda_{N_{\tilde{M}(t)}}\delta_{\tilde{M}(t)} + \lambda_{N_{\tilde{S}(t)}}\delta_{\tilde{S}(t)}\right]\|x_n(t) - x_{n-1}(t)\|. \tag{17}$$

By using Theorem 1 and $k(t)$-strongly accretiveness of $\tilde{T}$ with respect to g, we have

$$\begin{aligned}
\|x_n(t)-x_{n-1}(t)\|^2 &= \|J^{\partial_\eta\phi}_{\rho(t)}(z_n(t))-J^{\partial_\eta\phi}_{\rho(t)}(z_{n-1}(t)) \\
&\quad -[g(t,w_n(t))-x_n(t)-g(t,w_{n-1}(t))+x_{n-1}(t)]\|^2 \\
&\le \|J^{\partial_\eta\phi}_{\rho(t)}(z_n(t))-J^{\partial_\eta\phi}_{\rho(t)}(z_{n-1}(t))\|^2-2\langle g(t,w_n(t))-x_n(t) \\
&\quad -(g(t,w_{n-1}(t))-x_{n-1}(t)),j(x_n(t)-x_{n-1}(t))\rangle \\
&\le \frac{\tau^2(t)}{\alpha^2(t)}\|z_n(t)-z_{n-1}(t)\|^2-2k(t)\|x_n(t)-x_{n-1}(t)\|^2,
\end{aligned}$$

which implies that

$$\|x_n(t)-x_{n-1}(t)\| \le \frac{\tau(t)}{\alpha(t)\sqrt{1+2k(t)}}\|z_n(t)-z_{n-1}(t)\|. \tag{18}$$

Using (18), (17) becomes

$$\|z_{n+1}(t)-z_n(t)\| \le \frac{\left[\lambda_{J(t)}\lambda_{g(t)}\delta_{T(t)}+\lambda_{N_{\tilde{M}(t)}}\delta_{\tilde{M}(t)}+\lambda_{N_{\tilde{S}(t)}}\delta_{\tilde{S}(t)}\right]\tau(t)}{\alpha(t)\sqrt{1+2k(t)}}\|z_n(t)-z_{n-1}(t)\|,$$

i.e.,

$$\|z_{n+1}(t)-z_n(t)\| \le \Theta(t)\|z_n(t)-z_{n-1}(t)\|, \tag{19}$$

where

$$\Theta(t)=\frac{\left[\lambda_{J(t)}\lambda_{g(t)}\delta_{T(t)}+\lambda_{N_{\tilde{M}(t)}}\delta_{\tilde{M}(t)}+\lambda_{N_{\tilde{S}(t)}}\delta_{\tilde{S}(t)}\right]\tau(t)}{\alpha(t)\sqrt{1+2k(t)}}. \tag{20}$$

It follows from (13) that $\Theta(t)<1$ for all $t\in\Omega$. Therefore $\{z_n(t)\}$ is a Cauchy sequence in E. Since E is complete, there exists a measurable mapping $z:\Omega\to E$ such that $z_n(t)\to z(t)$, for all $t\in\Omega$. From Algorithm 3.1, D-Lipschitz continuity of $\tilde{M}$, $\tilde{S}$ and $\tilde{T}$, we have

$$\begin{aligned}
\|u_n(t)-u_{n+1}(t)\| &\le D\big(\tilde{M}(t,x_n(t)),\tilde{M}(t,x_{n+1}(t))\big) \le \delta_{\tilde{M}(t)}\|x_n(t)-x_{n+1}(t)\|, \\
\|v_n(t)-v_{n+1}(t)\| &\le D\big(\tilde{S}(t,x_n(t)),\tilde{S}(t,x_{n+1}(t))\big) \le \delta_{\tilde{S}(t)}\|x_n(t)-x_{n+1}(t)\|, \\
\|w_n(t)-w_{n+1}(t)\| &\le D\big(\tilde{T}(t,x_n(t)),\tilde{T}(t,x_{n+1}(t))\big) \le \delta_{\tilde{T}(t)}\|x_n(t)-x_{n+1}(t)\|,
\end{aligned}$$

which imply that $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in E. Let $u_n(t)\to u(t)$, $v_n(t)\to v(t)$ and $w_n(t)\to w(t)$. Since $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are Cauchy sequences of measurable mappings, we know that $u,v,w:\Omega\to E$ are also measurable.

Now, we will prove that $u(t)\in\tilde{M}(t,x(t))$, $v(t)\in\tilde{S}(t,x(t))$ and $w(t)\in\tilde{T}(t,x(t))$. For any $t\in\Omega$, we have

$$\begin{aligned}
d(u(t),\tilde{M}(t,x(t))) &\le \|u(t)-u_n(t)\|+d(u_{n-1}(t),\tilde{M}(t,x(t)) \\
&\le \|u(t)-u_n(t)\|+D(\tilde{M}(t,x_n(t)),\tilde{M}(t,x(t))) \\
&\le \|u(t)-u_n(t)\|+\delta_{\tilde{M}(t)}\|x_n(t)-x(t)\|\to 0.
\end{aligned}$$

Hence $u(t)\in\tilde{M}(t,x(t))$, for all $t\in\Omega$. Similarly, we can show that $v(t)\in\tilde{S}(t,x(t))$ and $w(t)\in\tilde{T}(t,x(t))$, for all $t\in\Omega$. This completes the proof. □

4. Conclusion

Due to tremendous applications of fuzzy random theory, in this work we introduce and study a random proximal operator equation for random fuzzy mappings in Banach spaces. We have shown that the random proximal operator equation for random fuzzy mappings is equivalent to a random variational-like inclusion problem for random fuzzy mappings and random variational-like inclusion problem for random fuzzy mapping is equivalent to a random fixed point problem. This equivalence is used to define a random iterative algorithm for solving random proximal operator equation for random fuzzy mappings.

By using the concept of 0-DQCV mappings, we establish an existence and convergence result for random proximal operator equation for random fuzzy mappings. We also compare random variational-like inclusion problem for random fuzzy mapping with random Wardrop equilibrium problem.

REFERENCES

Agarwal, R.P., Cho, Y.J. and Huang, N.,-J. (2000). Sensitivity analysis for strongly nonlinear quasi-variational inclusions, Appl. Math. Lett., Vol. 13, No. 6, pp. 19-24.

Agarwal, R.P., Huang, N.,-J. and Cho, Y.J. (2002). Generalized nonlinear mixed implicit quasi-variational inclusions with set-valued mappings, J. Inequal. Appl., Vol. 7, No. 6, pp. 807-828.

Ahmad, R., Siddiqi, A.H., and Khan, Z. (2005). Proximal point algorithm for generalized multivalued nonlinear quasi-variational-like inclusions in Banach spaces, Appl. Math. Comput., Vol. 163, pp. 295-308.

Ahmad, R. and Bazán, F.F. (2005). An iterative algorithm for random generalized nonlinear mixed variational inclusions for random fuzzy mappings, Appl. Math. Comput., Vol. 167, pp. 1400-1411.

Ahmad, R. and Farajzadeh, A.P. (2009). On random variational inclusions with random fuzzy mappings and random relaxed cocoercive mappings, Fuzzy Sets and Systems, Vol. 160, pp. 3166-3174.

Ahmad, R., Rahaman, M. and Rizvi, H.A. (2015). The fuzzy over-relaxed proximal point iterative scheme for generalized variational inclusion with fuzzy mappings, Applications and Applied Mathematics, Vol. 10, No. 2, pp. 805-816.

Ansari, Q.H. (1988). *Certain problems concerning variational inequalities*, Ph.D Thesis, A.M.U. Aligarh, India.

Chang, S.S. and Huang, N.-J. (1993). Generalized complementarity problem for fuzzy mappings, Fuzzy Sets and Systems, Vol. 55, pp. 227-234.

Chang, S.S. and Zhu, Y. (1989). On variational inequalities for fuzzy mappings, Fuzzy Sets and Systems, Vol. 32, pp. 356-367.

Chang, S.S. and Huang, N.-J. (1993). Generalized random multivalued quasi- complementarity problems, Ind. J. Math., Vol. 35, pp. 305-320.

Cho, Y.J., Huang, N.-J. and Kang, S.M. (2000). Random generalized set-valued strongly nonlinear implicit quasi-variational inequalities, J. Inequal. Appl., Vol. 5, pp. 531-575.

Ding, X.-P. and Park, J.-Y. (2002). A new class of generalized nonlinear implicit quasi-variational

inclusions with fuzzy mappings, J. Comput. Appl. Math., Vol. 138, pp. 243-257.

Ding, X.-P. and Lou, C.L (2000). Perturbed proximal point algorithms for general quasi-variational-like inclusions, J. Comut. Appl. Math., Vol. 210, pp. 153-165.

Deepmala, (2014). *A study on fixed point theorems for nonlinear contractions and its applications*. Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India-492010.

Farajzadeh, A.P., Mursaleen, M. and Shafie, A. (2016). On mixed vector equilibrium problems, Azerbaijan Jour. Math., Vol. 6, No. 2, pp. 87-102.

Gwinner, J. and Raciti, F. (2010). *Random variational inequalities with applications to equilibrium problems under uncertainty, Modeling Simulation and Optimization-Tolerance and Optimal Control*, Shkelzen Cakaj (Ed.).

Huang, N.-J. and Cho, Y.-J. (1999). Random completely generalized set-valued implicit quasi-variational inequalities, Positivity, Vol. 3, pp. 201-2013.

Huang, N.-J. and Lan, H.-Y. (2005). A couple of nonlinear equations with fuzzy mappings in fuzzy normed spaces, Fuzzy Sets and Systems, Vol. 152, pp. 209-222.

Himmelberg, C.J. (1975). Measurable relations, Fund. Math., Vol. 87, pp. 53-75.

Husain, S., Gupta, S. and Mishra, V.N. (2013). Graph convergence for the $H(.,.)$-mixed mapping with an application for solving the system of generalized variational inclusions, Fixed Point Theory Appl., Vol. 2013, pp. 304, doi:10.1186/10.1186/1687-1812-2013-304.

Husain, S., Gupta, S. and Mishra, V.N. (2013). An existence theorem of solutions for the system of generalized vector quasi-variational inequalities, Ameri. J. Oper. Res. (AJOR), Vol. 2013, No. 3, pp. 329-336, doi:10.4236/ajor.2013.33029.

Jin, M.-M. and Liu, Q.-K. (2004). Nonlinear quasi-variational inclusions involving generalized m-accretive mappings, Nonlinear Funct. Anal. Appl., Vol. 9, No. 3, pp. 485-494.

Lan, H.-Y. and Verma, R.U. (2008). Iterative algorithms for nonlinear fuzzy variational inclusions systems with (A, η)-accerative mappings in Banach spaces, Adv. Nonlinear Var. Inequal., Vol. 11, No. 1, pp. 15-30.

Lan, H.-Y., Kim, J.K. and Huang, N.-J. (2004). On the generalized nonlinear quasi-variational inclusions involving non-monotone set-valued mappings, Nonlinear Funct. Anal. Appl., Vol. 9, No. 3, pp. 451-465.

Lan, H.-Y., Liu, Q.-K. and Li, J. (2004). Iterative approximation for a system of nonlinear variational inclusions involving generalized m-accretive mappings, Nonlinear Anal. Forum, Vol. 9, No. 1, pp. 33-42.

Lan, H.-Y., He, Z.-Q. and Li, J. (2003). Generalized nonlinear fuzzy quasi-variational-like inclusions involving maximal η-monotone mappings, Nonlinear Anal. Forum, Vol. 8, No. 1, pp. 43-54.

Lan, H.-Y. (2006). Projection iterative approximations for a new class of general random implicit quasi-variational inequalities, J. Inequal. Appl., Vol. 27, Article ID 1081261.

Lan, H.-Y., Cho, Y.J. and Xie, W. (2009). General nonlinear random equations with random multivalued operator in Banach spaces, J. Inequal. Appl., Vol. 17, Article ID 865093.

Liu, L.-W. and Li, Y.-Q. (2001). On generalized set-valued variational inclusions, J. Math. Anal. Appl., Vol. 261, No. 1, pp. 231-240.

Mishra, V.N. (2007). *Some problems on approximations of functions in Banach spaces*. Ph.D. Thesis, Indian Institute of Technology, Roorkee-247667 Uttarakhand.

Murthy, P.P., Rashmi and Mishra, V.N. (2016). Tripled coincidence point theorem for compatible maps in fuzzy metric spaces, Elect. J. Math. Anal., Vol. 4, No. 2, pp. 96-106.

Noor, M.A. (1989). Variational inequalities with fuzzy mappings (I), Fuzzy Sets and Systems, Vol. 55, pp. 309-314.

Onjai-Uea, N. and Kumam, P. (2010). A generalized nonlinear random equations with random fuzzy mappings in uniformly smooth Banach spaces, J. Inequl. Appl., Article ID 728452, doi:10.1155/2010/728452.

Park, J.-Y. and Jeong, J.-U. (2000). Iterative algorithm for finding approximate solutions to completely generalized strongly quasi-variational inequalities for fuzzy mapping, Fuzzy Sets and Systems, Vol. 115, pp. 413-418.

Park, J.-Y. and Jeong, J.-U. (2000). A perturbed algorithm of variational inclusions for fuzzy mappings, Fuzzy Sets and Systems, Vol. 115, pp. 419-424.

Verma, R.U., Khan, M.F. and Salahuddin (2005). Generalized set-valued nonlinear mixed quasi-variational-like inclusions with fuzzy mappings, Adv. Nonlinear Var. Inequal., Vol. 8, No. 2, pp. 11-37.

Zadeh, L.A. (1965). Fuzzy Sets, Information and Control, Vol. 8, pp. 338-353.

Zimmermann, H.J. (1988). *Fuzzy set theory and its applications*, Kluwer Acad. Publ., Dordrecht.