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On the convergence of two-dimensional fuzzy Volterra-Fredholm integral equations by using Picard method

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Abstract

In this paper we prove convergence of the method of successive approximations used to approximate the solution of nonlinear two-dimensional Volterra-Fredholm integral equations and define the notion of numerical stability of the algorithm with respect to the choice of the first iteration. Also we present an iterative procedure to solve such equations. Finally, the method is illustrated by solving some examples.

Keywords: Convergence; Picard method; Volterra-Fredholm integral equations; Unique analysis

MSC 2010 No.: 34A12, 26A33

1. Introduction

In recent years topics of fuzzy integral equations have been developed and have attracted growing interest, in particular in relation to fuzzy control. Fuzzy equations are used to model uncertainty mathematical models. Mathematical models used in many problems of physics, biology, chemistry, engineering and in other areas are based on these equations (Puri et al., 1986; Gasilova et al., 2014).

The main problems that arise out of fuzzy integral equations are: existence and uniqueness of the solution, and the construction of numerical methods of approximation. Fixed point theorems like

Darbo's theorem and Banach's fixed point principle are powerful tools used to prove existence and uniqueness of the solution for fuzzy integral equations (Balachandran and Prakash, 2002; She, 2016; Balachandran et al., 2005; Mihai et al., 2015). Most of these theorems are based on compactness of operators on a suitable Banach space. In this study use the function $f(s,t)$ satisfying the Lipschitz condition.

Many real-world engineering and mechanics problems can be modeled as two-dimensional fuzzy integral equations. It is important to develop convergence and unique analysis for fuzzy integral equations. Much research has been undertaken on analyzing existence and uniqueness of solution and developing numerical algorithms for solving one-dimensional integral equations (Gonga et al., 2015; Macarioa and Sanchis, 2015; Tron Thuana et al., 2014; Behzadi et al., 2012), but in two-dimensional cases very little has been done (Sadatrasoul and Ezzati, 2016).

In the present article, in order to obtain the convergence and uniqueness of solution for two-dimensional fuzzy Volterra- Fredholm integral equations (2D-FVFIE), we use the Picard method to solve 2D-FVFIE of the form:

$$\begin{aligned}\tilde{f}(s,t) = \quad & \tilde{g}(s,t) + \lambda_1 \int_0^t \int_0^s U_1(s,t,x,y,\tilde{f}(x,y))dxdy \\ & +\lambda_2 \int_0^{T_2} \int_0^{T_1} U_2(s,t,x,y,\tilde{f}(x,y))dxdy,\end{aligned} \tag{1}$$

and

$$\tilde{f}(s,t) = \tilde{g}(s,t) + \int_0^s \int_\Omega U(s,t,x,y,\tilde{f}(x,y))dydx, \quad D = [0,T] \times \Omega, \tag{2}$$

where $\tilde{f}(s,t)$ is an unknown function which should be determined, and Ω is a closed subset of R^n, $n = 1,2,3$. The functions $U_1(s,t,x,y,\tilde{f})$ and $U_2(s,t,x,y,\tilde{f})$ are given functions defined on

$$W = \{(s,t,x,y,\tilde{f}) : 0 \le x \le s < T_1, 0 \le y \le t < T_2\},$$

$$S = \{(s,t,x,y,\tilde{f}) : 0 \le x \le s < T, t \in \Omega, y \in \Omega\}.$$

For convenience, we put

$$U_1(s,t,x,y,\tilde{f}) = k_1(s,t,x,y)[\tilde{f}(s,t)]^{p_1},$$

$$U_2(s,t,x,y,\tilde{f}) = k_2(s,t,x,y)[\tilde{f}(s,t)]^{p_2},$$

where p_1 and p_2 are positive integers. Moreover $\tilde{g}(s,t)$ is a known function defined on D. Since any finite interval $[a,b]$ can be transformed to $[0,1]$ by linear maps, it is supported that $[0,T_1) = (0,T_2) = [0,1)$, without any loss of generality.

2. Preliminaries

An arbitrary fuzzy number with an ordered pair of functions $(\underline{u}(r), \bar{u}(r)), \quad 0 \leq r \leq 1$, which satisfy the following requirements is represented (Mihai and Popescu, 2015).

Definition 1.

A fuzzy number is a function $u : R \to [0, 1]$ satisfying the following properties:

(a) u is upper semicontinuous on R,
(b) $u(x) = 0$ outside of some interval $[c, d]$,
(c) there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c, a]$, decreasing on $[b, d]$ and $u(x) = 1$ for each $x \in [a, b]$, and
(d) u is fuzzy convex set (that is $u(\lambda x + (1 - \lambda)y) \geq min\,\{u(x), u(y)\}$, $\forall x, y \in R, \lambda \in [0, 1]$).

The set of all fuzzy numbers is denoted by R_F. Any real number $a \in R$ can be interpreted as a fuzzy number $a = \chi_{\{a\}}$ and therefore $R \subset R_F$. Also, the neutral element with respect to $\oplus$ in R_F is denoted by $\tilde{0} = \chi_{\{0\}}$.

A fuzzy number u in parametric form is a pair $(\underline{u}, \bar{u})$, of functions $\underline{u}(r), \bar{u}(r)$, $0 \leq \alpha \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,
2. $\bar{u}(r)$ is a bounded non-increasing left continuous function in $(0, 1]$, and right continuous at 0,
3. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

Let E be the set of all upper semicontinuous normal convex fuzzy numbers with bounded *r*-level intervals. This means that if $\nu \in E$, then the r-level set

$$[\nu]^r = \{s|\nu(s) \geq r\}, 0 < r \leq 1,$$

is a closed bounded interval which is denoted by

$$[\nu]^r = [\underline{\nu}(r), \bar{\nu}(r)]. \tag{3}$$

For arbitrary $u = (\underline{u}, \bar{u})$, $\nu = (\underline{\nu}, \bar{\nu})$ and $k \geq 0$, addition $(u + \nu)$ and multiplication by k as $(\underline{u + \nu}) = \underline{u}(r) + \underline{\nu}(r)$, $(\overline{u + \nu}) = \bar{u}(r) + \bar{\nu}(r)$, $(\underline{ku}(r)) = k\underline{u}(r)$, $\overline{ku}(r) = k\bar{u}(r)$ are defined.

Each $y \in R$ can be regarded as a fuzzy number y defined by

$$\tilde{y}(k) = \begin{cases} 1 & t = y\,, \\ 0 & t \neq y\,. \end{cases}$$

Definition 2.

The Hausdorff distance between fuzzy numbers is given by $D : E \times E \to R_+ \cup \{0\}$,

$$D(u,v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}. \tag{4}$$

It is easy to see that D is a metric in E and has the following properties (Puri et al., 1986).

(a) $D(u \oplus w, \nu \oplus w) = D(u, \nu), \forall u, \nu, w \in E,$
(b) $D(k \odot u, k \odot \nu) = |k| D(u, \nu), \forall \in R, u, \nu \in E,$
(c) $D(u \oplus \nu, w \oplus e) \leq D(u, w) + D(\nu, e), \forall u, \nu, w, e \in E,$
(d) (D, E) is a complete metric space.

Definition 3.

Let $f : R \to E$ be a fuzzy valued function. If for arbitrary fixed $t_0 \in R$ and $\varepsilon > 0$, exists $\delta > 0$ such that

$$\mid t - t_0 \mid < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon,$$

f is said to be continuous (Salahshour and Allahviranloo, 2012).

Theorem 1.

Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q), p, q > 0$, $y_0 \in R_f$ and $f : R_0 \to R_f$ be continuous such that $\|f(x,y)\|_f \leq M$ for all $(x,y) \in R_0$ and f satisfies the Lipschitz condition $D(f(x,y), f(x,z)) \leq L.\, D(y,z), \forall (x,y), (x,z) \in R_0$ and $D(y,z) \leq q$. If there exists $d > 0$ such that for $x \in (x_0, x_0+d)$ the sequence given by $\overline{y_0}(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 - (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt$ is defined for any $n \in N$, then the fuzzy initial value problem $\acute{y} = f(x,y), y(x_0) = y_0$ has two solutions $y, \overline{y} : [x_0, x_0 + r] \to \overline{B}(y_0, q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ and the successive iterations in

$$\begin{cases} y_0(x) = y_0, \\ y_{n+1}(x) = y_0 \oplus \int_{x_0}^{x} f(t, y_n(t))dt, \end{cases}$$

and

$$\begin{cases} \overline{y}_0(x) = y_0, \\ \overline{y}_{n+1}(x) = y_0 - (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt, \end{cases}$$

converge to these two solutions, respectively (Bede and Gal, 2005).

3. Convergence and unique analysis

In this section, we prove the existence and uniqueness of the solution and convergence of the method by using the following assumptions.

We can write successive iterations (by using Picard method) as follows:

$$\begin{aligned} \tilde{f}_{n+1}(s,t) = \quad & \tilde{g}(s,t) + \lambda_1 \int_0^t \int_0^s k_1(s,t,x,y)[\tilde{f}_n(s,t)]^{P_1} dxdy \\ & + \lambda_2 \int_0^1 \int_0^1 k_2(s,t,x,y)[\tilde{f}_n(s,t)]^{P_2} dxdy, \end{aligned} \tag{5}$$

and

$$\tilde{f}_{n+1}(s,t) = \tilde{g}(s,t) + \int_0^s \int_0^1 k(s,t,x,y)[\tilde{f}_n(s,t)]^P dydx. \tag{6}$$

Let $\tilde{f}(s,t), \tilde{g}(s,t) : [0,T_1] \times [0,T_2] \to R_F$ be continuous functions.

$$D(\tilde{f}_1, \tilde{f}_2) = \sup \left\{ |\tilde{f}_1(s,t) - \tilde{f}_2(s,t)|, (s,t) \in [0,T_1] \times [0,T_2] \right\} \; 0 \leq t < T_2, \; 0 \leq s < T_1. \tag{7}$$

Also, we suppose the nonlinear operators $\tilde{f}(s,t)^{p_1}$ and $\tilde{f}(s,t)^{p_2}$ satisfy Lipschitz conditions with

$$\begin{aligned} |(\tilde{f}_1(s,t))^{p_1} - (\tilde{f}_2(s,t))^{p_1}| &= |\tilde{f}_1(s,t) - \tilde{f}_2(s,t)||(\tilde{f}_1(s,t))^{p_1-1} \\ &\quad +(\tilde{f}_1(s,t))^{p_1-2}(\tilde{f}_2(s,t)) \ldots + f_2(\tilde{s},t)|. \end{aligned}$$

Now we have $L_1^p > 0$

$$|(\tilde{f}_1(s,t))^{p_1-1} + (\tilde{f}_1(s,t))^{p_1-2}(\tilde{f}_2(s,t))^1 + \ldots + \tilde{f}_2(s,t)| \leqslant L_1^{p_1} \qquad \exists L_1 > 0.$$

Thus,

$$| \; \tilde{f}_1(s,t))^{p_1} - (\tilde{f}_2(s,t))^{p_1} \; | \leqslant L_1^{P_1} |\tilde{f}1(s,t) - \tilde{f}_2(s,t)|.$$

After applying the sup, we have

$$\sup |(\tilde{f}_1(s,t))^{p_1} - (\tilde{f}_2(s,t))^{p_1}| \leqslant L_1^{p_1} \sup |\tilde{f}_1(s,t) - \tilde{f}_2(s,t)|.$$

Using equation (7), then it is easy to write

$$\begin{aligned} D(\tilde{f}_1(s,t)^{p_1}, \tilde{f}_2(s,t)^{p_1}) &\leqslant L_1^{p_1} D(\tilde{f}_1(s,t), \tilde{f}_2(s,t)), \quad L_1 > 0, \\ D(\tilde{f}_1(s,t)^{p_2}, \tilde{f}_2(s,t)^{p_2}) &\leqslant L_2^{P_2} D(\tilde{f}_1(s,t), \tilde{f}_2(s,t)), \quad L_2 > 0, \\ \exists M_1 > 0 \qquad s.t & \qquad |\lambda_1 k_1(s,t,x,y)| \leqslant M_1, \\ \exists M_2 > 0 \qquad s.t & \qquad |\lambda_2 k_2(s,t,x,y)| \leqslant M_2. \end{aligned}$$

Then,

$$\alpha = (M_1 L_1^P + M_2 L_2^{p_2}).$$

The proof of equation (2) is similar.

$$D(\tilde{f}_1(s,t)^p, \tilde{f}_2(s,t)^p) \leqslant L^p D(\tilde{f}_1(s,t), \tilde{f}_2(s,t)), \quad L > 0.$$

Let $0 < \alpha < 1$ and $\exists M > 0$ s.t $| \; k(s,t,x,y) \; | \leq M$ then $\alpha = ML^p$.

Theorem 2.

Equations (1) and (2) have unique solution when $0 < \alpha \leq 1$, respectively.

Proof:

Let $\tilde{f}_1(s,t)$ and $\tilde{f}_2(s,t)$ be two different solutions of equation (1),

$$
\begin{aligned}
D(\tilde{f}_1(s,t),\tilde{f}_2(s,t)) &= D(\tilde{g}(s,t)+\lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_1(s,t))^{p_1}dxdy \\
&\quad +\lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_1(s,t))^{p_2}dxdy, \\
&\quad \tilde{g}(s,t)+\lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_2(s,t))^{p_1}dxdy \\
&\quad +\lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_2(s,t))^{p_2}dxdy) \\
&= D(\lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_1(s,t))^{p_1}dxdy \\
&\quad +\lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_1(s,t))^{p_2}dxdy, \\
&\quad \lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_2(s,t))^{p_1}dxdy \\
&\quad +\lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_2(s,t))^{p_2}dxdy) \\
&\leqslant D(\lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_1(s,t))^{p_1}dxdy, \\
&\quad \lambda_1\int_0^t\int_0^s K_1(s,t,x,y)(\tilde{f}_2(s,t))^{p_1}dxdy) \\
&\quad +D(\lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_1(s,t))^{p_2}dxdy, \\
&\quad \lambda_2\int_0^1\int_0^1 K_2(s,t,x,y)(\tilde{f}_2(s,t))^{p_2}dxdy) \\
&\leqslant (M_1L_1^{p_1}+M_2L_2^{p_2})D(\tilde{f}_1(s,t),\tilde{f}_2(s,t)) \\
&= \alpha D(\tilde{f}_1(s,t)),\tilde{f}_2(s,t)),
\end{aligned}
$$

from which we get $(1-\alpha)D(\tilde{f}_1(s,t)),\tilde{f}_2(s,t))\leq 0$. Since $0<\alpha<1$, then $D(\tilde{f}_1(s,t)),\tilde{f}_2(s,t))=0$ implies $\tilde{f}_1(s,t)=\tilde{f}_2(s,t)$.

Now for equation (2), we have

$$
\begin{aligned}
D(\tilde{f}_1(s,t),\tilde{f}_2(s,t)) &= D(\tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,s,y)(\tilde{f}_1(s,t))^{p}dydx, \\
&\quad \tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,s,y)(\tilde{f}_2(s,t))^{p}dydx) \\
&= D\left(\int_0^s\int_0^1 k(s,t,x,y)(\tilde{f}_1(s,t))^{p}dydx,\int_0^s\int_0^1 k(s,t,x,y)(\tilde{f}_2(s,t))^{p}dydx\right) \\
&\leq ML^P D(\tilde{f}_1(s,t),\tilde{f}_2(s,t))=\alpha D(\tilde{f}_1(s,t),\tilde{f}_2(s,t)),
\end{aligned}
$$

from which we get $(1-\alpha)D(\tilde{f}_1(s,t)),\tilde{f}_2(s,t))$. Since $0<\alpha<1$, then $D(\tilde{f}_1(s,t)),\tilde{f}_2(s,t))=0$ implies $\tilde{f}_1(s,t)=\tilde{f}_2(s,t)$ and completes the proof. □

Theorem 3.

The solution $\tilde{f}_n(s,t)$ by using Picard method converges to the exact solution of the equations (1) and (2), $0 < \alpha \leq 1$.

Proof:

We prove equation (1) as follows:

$$\begin{aligned}
D(\tilde{f}_{n+1}(s,t), \tilde{f}(s,t)) &= D(\tilde{g}(s,t) + \lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}_n(s,t))^{p_1} dxdy \\
&\quad +\lambda_2 \int_0^1 \int_0^1 K_2(s,t,x,y)(\tilde{f}_n(s,t))^{p_2} dxdy, \\
&\quad \tilde{g}(s,t) + \lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}(s,t))^{p_1} dxdy \\
&\quad +\lambda_2 \int_0^1 \int_0^1 K_2(s,t,x,y)(\tilde{f}(s,t))^{p_2} dxdy) \\
&= D(\lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}_n(s,t))^{p_1} dxdy \\
&\quad +\lambda_2 \int_0^1 \int_0^1 K_2(s,t,x,y)(\tilde{f}_n(s,t))^{p_2} dxdy, \\
&\quad \lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}(s,t))^{p_1} dxdy \\
&\quad +\lambda_2 \int_0^1 \int_0^1 K_2(s,t,x,y)(\tilde{f}(s,t))^{p_2} dxdy) \\
&\leqslant D(\lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}_n(s,t))^{p_1} dxdy, \\
&\quad \lambda_1 \int_0^t \int_0^s K_1(s,t,x,y)(\tilde{f}(s,t))^{p_1} dxdy) + \\
&\quad D(\lambda_2 \int_0^1 \int_0^1 K_2(s,tx,y)(\tilde{f}_n(s,t))^{p_2} dxdy, \\
&\quad \lambda_2 \int_0^1 \int_0^1 K_2(s,t,x,y)(\tilde{f}(s,t))^{p_2} dxdy) \\
&\leqslant (M_1 L_1^{p_1} + M_2 L_2^{p_2}) D(\tilde{f}_n(s,t), \tilde{f}(s,t)) \\
&= \alpha D(\tilde{f}_n(s,t), \tilde{f}(s,t)).
\end{aligned}$$

Since $0 < \alpha < 1$, then $D(\tilde{f}_n(s,t), \tilde{f}(s,t)) \to 0$ as $n \to \infty$. Therefore, $\tilde{f}_n(s,t) \to \tilde{f}(s,t)$.

The proof of equation (2) is similar to previous theorems.

We have:

$$
\begin{aligned}
D(\tilde{f}_{n+1}(s,t),\tilde{f}(s,t)) &= D(\tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,s,y)(\tilde{f}_n(s,t))^p dydx,\\
&\quad \tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,s,y)(\tilde{f}(s,t))^p dydx)\\
&= D\left(\int_0^s\int_0^1 k(s,t,x,y)(\tilde{f}_n(s,t))^p dydx, \int_0^s\int_0^1 k(s,t,x,y)(\tilde{f}(s,t))^p dydx\right)\\
&\le ML^P D(\tilde{f}_n(s,t),\tilde{f}(s,t)) = \alpha D(\tilde{f}_n(s,t),\tilde{f}(s,t)).
\end{aligned}
$$

Since $0<\alpha<1$, then $D(\tilde{f}_n(s,t),\tilde{f}(s,t)) \to 0$ as $n\to\infty$. Therefore, $\tilde{f}_n(s,t)\to\tilde{f}(s,t)$. □

4. Solving 2D-FVFIE

In this section, we present an effective method for solving 2D-FVFIE by using the Picard method. Consider the following 2D-FVFIE:

$$
\begin{aligned}
\tilde{f}(s,t) = \; & \tilde{g}(s,t)+\lambda_1\int_0^t\int_0^s k_1(s,t,x,y)[\tilde{f}(x,y)]^{p_1}dxdy\\
& +\lambda_2\int_0^1\int_0^1 k_2(s,t,x,y)[\tilde{f}(x,y)]^{p_2}dxdy,
\end{aligned} \tag{8}
$$

and

$$
\tilde{f}(s,t)=\tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,x,y)[\tilde{f}(x,y)]^p dydx. \tag{9}
$$

Now, we can write successive iterations (by using the Picard method) as follows:

$$
\begin{aligned}
\tilde{f}_{n+1}(s,t) = \; & \tilde{g}(s,t)+\lambda_1\int_0^t\int_0^s k_1(s,t,x,y)[\tilde{f}_n(x,y)]^{p_1}dxdy\\
& +\lambda_2\int_0^1\int_0^1 k_2(s,t,x,y)[\tilde{f}_n(x,y)]^{p_2}dxdy,
\end{aligned} \tag{10}
$$

and

$$
\tilde{f}_{n+1}(s,t)=\tilde{g}(s,t)+\int_0^s\int_0^1 k(s,t,x,y)[\tilde{f}_n(x,y)]^p dydx. \tag{11}
$$

We present the Picard method in several steps.

Step 1. Set $n\to 0$

Step 2. Calculate the recursive relations (Bede and Gal, 2005; Mikaeilvand and Khakrangin,

2012) and (Bede and Gal, 2005; Gasilova et al., 2014).

Step 3. If $D(\tilde{f}_{n+1}, \tilde{f}_n) < \epsilon$, where ϵ is a given positive value then go to step 4, else $n \to n+1$ and go to step 2.

Step 4. Print $\tilde{f}_n$ as the approximation of the exact solution.

Since in this paper we consider fuzzy integral equations, according to Definition 1 equation (Bede and Gal, 2005; Behzadi et al., 2012) and (Bede and Gal, 2005; Salahshour and Allahviranloo, 2012) can be represented in the form $\tilde{f}(s,t) = (\underline{\tilde{f}}(s,t), \overline{\tilde{f}}(s,t))$ of functions $\underline{\tilde{f}}(s,t,r), \overline{\tilde{f}}(s,t,r)$, $0 \le r \le 1$, where

$$\begin{aligned}
\underline{\tilde{f}}(s,t,r) &= \underline{\tilde{g}}(s,t,r) + \lambda_1 \int_0^t \int_0^s k_1(s,t,x,y)[\underline{\tilde{f}}(x,y,r)]^{p_1} dxdy \\
&\quad + \lambda_2 \int_0^1 \int_0^1 k_2(s,t,x,y)[\underline{\tilde{f}}(x,y,r)]^{p_2} dxdy, \\
\overline{\tilde{f}}(s,t,r) &= \overline{\tilde{g}}(s,t,r) + \lambda_1 \int_0^t \int_0^s k_1(s,t,x,y)[\overline{\tilde{f}}(x,y,r)]^{p_1} dxdy \\
&\quad + \lambda_2 \int_0^1 \int_0^1 k_2(s,t,x,y)[\overline{\tilde{f}}(x,y,r)]^{p_2} dxdy.
\end{aligned}$$

Also equation (Bede and Gal, 2005; Salahshour and Allahviranloo, 2012) can be written as

$$\underline{\tilde{f}}(s,t,r) = \underline{\tilde{g}}(s,t,r) + \int_0^s \int_0^1 k(s,t,x,y)[\underline{\tilde{f}}(x,y,r)]^p dydx,$$

and

$$\overline{\tilde{f}}(s,t,r) = \overline{\tilde{g}}(s,t,r) + \int_0^s \int_0^1 k(s,t,x,y)[\overline{\tilde{f}}(x,y,r)]^p dydx.$$

5. Numerical examples

In order to illustrate the effectiveness of the method proposed in this paper, two numerical examples are presented.

Example 1.

Consider the following fuzzy Volterra-Fredholm integral equation

$$\tilde{f}_{n+1}(s,t) = \tilde{g}(s,t) - \int_0^s \int_0^1 t^2 e^{-y} \tilde{f}_n(x,y) dydx, \quad 0 \le s < 1, \quad 0 \le t < 1.$$

The general form of the equation is

$$\underline{\tilde{f}}_{n+1}(s,t,r) = \underline{\tilde{g}}(s,t,r) - \int_0^s \int_0^1 t^2 e^{-y} \underline{\tilde{f}}_n(x,y,r)dydx,$$

and

$$\overline{\tilde{f}}_{n+1}(s,t,r) = \overline{\tilde{g}}(s,t,r) - \int_0^s \int_0^1 t^2 e^{-y} \overline{\tilde{f}}_n(x,y,r)dydx,$$

where

$$\begin{aligned} \underline{\tilde{g}}(s,t,r) &= s^2e^t - \frac{2}{3}s^3t^2(r^2+r), \\ \overline{\tilde{g}}(s,t,r) &= s^2e^t - \frac{2}{3}s^3t^2(4-r^3-r). \end{aligned}$$

The exact solution is:

$$\underline{\tilde{f}}_n(s,t,r) = s^2e^t(r^2+r),$$

and

$$\overline{\tilde{f}}_n(s,t,r) = s^2e^t(4-r^3-r).$$

Let

$$\underline{\tilde{f}}_0(s,t,r) = \underline{\tilde{g}}(s,t,r) = s^2e^t - \frac{2}{3}s^3t^2(r^2+r).$$

Setting $n \to 0$, we have

$$\begin{aligned} \underline{\tilde{f}_1}(s,t,r) &= s^2e^t - \frac{2}{3}s^3t^2(r^2+r) - \int_0^s \int_0^1 t^2e^{-y}\underline{\tilde{f}_0}(x,y,r)dxdy \\ &= (s^2e^t - s^3t^2 - \frac{5}{6}s^4t^2e^{-1} + \frac{1}{3}s^4t^2)(r^2+r). \end{aligned}$$

Setting $n \to 1$, we have:

$$\begin{aligned} \underline{\tilde{f}_2}(s,t,r) &= s^2e^t - \frac{2}{3}s^3t^2(r^2+r) - \int_0^s \int_0^1 t^2e^{-y}\underline{\tilde{f}_1}(x,y,r)dxdy \\ &= (s^2e^t - s^3t^2 + \frac{1}{2}s^4t^2 - \frac{2}{15}s^5t^2 - \frac{5}{4}s^4t^2e^{-1} - \frac{5}{6}s^5e^{-2} + \frac{1}{3}s^5t^2e^{-1} + \frac{1}{3}s^5e^{-1}) \\ &\quad \times(r^2+r). \\ &\vdots \end{aligned}$$

Similarly,

$$
\begin{aligned}
\overline{\tilde{f}}_1(s,t,r) &= (s^2e^t - s^3t^2 - \frac{5}{6}s^4t^2e^{-1} + \frac{1}{3}s^4t^2)(4 - r^3 - r),\\
\overline{\tilde{f}}_2(s,t,r) &= (s^2e^t - s^3t^2 + \frac{1}{2}s^4t^2 - \frac{2}{15}s^5t^2 - \frac{5}{4}s^4t^2e^{-1} - \frac{5}{6}s^5e^{-2} + \frac{1}{3}s^5t^2e^{-1} + \frac{1}{3}s^5e^{-1})\\
&\quad \times(4 - r^3 - r).\\
&\vdots
\end{aligned}
$$

If $D(\underline{\tilde{f}}_{n+1}, \underline{\tilde{f}}_n) < \epsilon$ and $D(\overline{\tilde{f}}_{n+1}, \overline{\tilde{f}}_n) < \epsilon$, where ϵ is a given positive value and $0 \le s < 1$, $0 \le t < 1$, then $\tilde{f}_n = (\underline{\tilde{f}}_n, \overline{\tilde{f}}_n)$ is the approximation of the exact solution, else $n \to n+1$ and calculate the recursive relation.

Example 2.

Consider the following two-dimensional fuzzy Volterra-Fredholm integral equation

$$
\tilde{f}_{n+1}(s,t) = \tilde{g}(s,t) - \int_0^s \int_0^1 (2y-1)e^x \tilde{f}_n(x,y)dydx,\ 0 \leqslant s < 1,\ 0 \leqslant t < 1,
$$

where

$$
\tilde{g}(s,t)(r) = \left(\sin s + t - \frac{1}{6}e^s + \frac{1}{6}(3+r), \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(5-r)\right).
$$

The exact solution is:

$$
\begin{aligned}
\underline{\tilde{f}}(s,t,r) &= \sin s + t - \frac{1}{3}e^s + \frac{1}{3}(3+r),\\
\overline{\tilde{f}}(s,t,r) &= \sin s + t - \frac{1}{3}e^s + \frac{1}{3}(5-r).
\end{aligned}
$$

This equation can be written as:

$$
\underline{\tilde{f}}_{n+1}(s,t,r) = \underline{\tilde{g}}(s,t,r) - \int_0^s \int_0^1 (2y-1)e^x \underline{\tilde{f}}_n(x,y,r)dydx,
$$

and

$$
\overline{\tilde{f}}_{n+1}(s,t,r) = \overline{\tilde{g}}(s,t,r) - \int_0^s \int_0^1 (2y-1)e^x \overline{\tilde{f}}_n(x,y,r)dydx,
$$

where

$$\underline{\tilde{g}}(s,t,r) = \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(3+r),$$

and

$$\bar{\tilde{g}}(s,t,r) = \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(5-r).$$

Let

$$\underline{\tilde{f}}_0(s,t,r) = \underline{\tilde{g}}(s,t,r) = \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(3+r).$$

Setting $n \longrightarrow 0$, we have:

$$\begin{aligned}
\underline{\tilde{f}}_1(s,t,r) &= \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(3+r) - \int_0^s \int_0^1 (2y-1)e^x(\sin x + y - \frac{1}{6}e^x + \frac{1}{6})(3+r)dx \\
&= (\sin s + t - \frac{1}{3}e^s + \frac{1}{3})(3+r).
\end{aligned}$$

Letting $n \longrightarrow 1$, then:

$$\begin{aligned}
\underline{\tilde{f}}_2(s,t,r) &= \sin s + t - \frac{1}{6}e^s + \frac{1}{6}(3+r) - \int_0^s \int_0^1 (2y-1)e^x(\sin x + y - \frac{1}{3}e^x + \frac{1}{3})(3+r)dydx \\
&= (\sin s + t - \frac{1}{3}e^s + \frac{1}{3})(3+r). \\
&\vdots
\end{aligned}$$

with the same procedure we can infer that the approximate solution

$$\underline{\tilde{f}}(s,t,r) = (\sin s + t - \frac{1}{3}e^s + \frac{1}{3})(3+r),$$

is the same as the exact solution.

Similarly,

$$\begin{aligned}
\bar{\tilde{f}}_1(s,t,r) &= (\sin s + t - \frac{1}{3}e^s + \frac{1}{3})(5-r), \\
\bar{\tilde{f}}_2(s,t,r) &= (\sin s + t - \frac{1}{3}e^s + \frac{1}{3})(5-r). \\
&\vdots
\end{aligned}$$

So, the approximate solution $\tilde{f}(s,t,r) = (\underline{\tilde{f}}(s,t,r), \bar{\tilde{f}}(s,t,r))$ is the same as the exact solution.

6. Conclusion

In this paper, the convergence and unique analysis of solutions for two-dimensional Volterra-Fredholm integral equations were studied. Two theorems for convergence and unique analysis of solutions were given and proved. Also the Picard method has been shown to solve effectively, easily, and accurately these equations.

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