Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

**Applications and Applied Mathematics:**
An International Journal
**(AAM)**

**Vol. 11, Issue 2 (December 2016), pp. 559 - 575**

# A 10-point Approximating Subdivision Scheme Based on Least Squares Technique

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Received: October 16, 2015; Accepted: May 24, 2016

## Abstract

In this paper, a 10-point approximating subdivision scheme is presented. Least squares technique for fitting the polynomial of degree 9 to data is used to develop this scheme. The proposed strategy can be used to generate a family of schemes. The important characteristics of the scheme are also discussed. Graphical efficiency of the scheme is shown by applying it on different types of data.

**Keywords:** Approximating subdivision scheme; least squares; regression; polynomial; curve fitting; continuity; local analysis; polynomial generation and reproduction

**MSC 2010 No.:** 93E24, 65D17, 65D07, 65D05, 65D10

## 1. Introduction

Subdivision scheme is widely used for curve and surface fitting from few decades. It is an algorithm to define smooth curves and surfaces as a sequence of successively refined control polygons. It has an abundant application in the field of science and engineering.

A general form of univariate binary subdivision scheme $S$ which maps polygon $f^k = \{f_i^k\}_{i\in\mathbb{Z}},\ k \geq 0$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbb{Z}}$ is defined in Dyn et al. (1991) by

$$\begin{cases} f_{2i}^{k+1} = \sum_{j\in\mathbb{Z}} \beta_{2j} f_{i-j}^k, \\ f_{2i+1}^{k+1} = \sum_{j\in\mathbb{Z}} \beta_{2j+1} f_{i-j}^k, \end{cases} \tag{1}$$

where the set $\beta = \{\beta_j : j \in \mathbb{Z}\}$ of coefficients is called mask/stencil of the subdivision scheme. The Laurent polynomial of the mask $\beta$ of scheme (1) is defined as $\beta(z) = \sum_{j\in\mathbb{Z}} \beta_j z^j$. A necessary condition for the uniform convergence of subdivision scheme defined in Equation (1) is $\sum_{j\in\mathbb{Z}} \beta_{2j} = 1, \sum_{j\in\mathbb{Z}} \beta_{2j+1} = 1.$ This is equivalent to $\beta(-1) = 0,\ \beta(1) = 2,$ which implies $\beta(z) = (1+z)b(z),\ b(1) = 1.$ For further reading regarding analysis of the scheme, see Hormann (2012) and Mustafa and Zahid (2013).

The subdivision schemes are different due to the different values of $\beta$'s. Several techniques have been used to find $\beta$'s. Deslauriers and Dubuc (1989) and Mustafa et al. (2014) used Lagrange polynomials to compute $\beta$'s while Lian (2009) used wavelet techniques for the computations of these values. Costantini and Manni (2010) used Hermite polynomials to generate the values of $\beta's$. One can get 2-point, 3-point,$\cdots n-$point schemes from the mask (i.e. the values of $\beta$'s) obtained from the above techniques. Romani (2015) introduced a class of subdivision schemes by making the variant of existing algorithms. Mustafa et al. (2016) introduced a family of schemes by convolving the existing schemes. A major advantage of the subdivision schemes is that they can be easily applied to virtually any data type. However, by Mustafa et al.(2015) early work in the subdivision schemes do not deal with noisy data with impulsive noises. Dyn et al. (2015) and Mustafa et al. (2015) pointed out that least squares based subdivision schemes are better choices to handle with these types of problems. Therefore, in this paper, we prefer to use least squares technique instead of other techniques.

The method of least squares is one of the golden techniques in statistics for curve fitting. In this modern era method of least squares is frequently used to find numerical values of the parameters to fit a function to set of data. It means that the overall solution minimizes the sum of the squares of the errors made in the results of every single equation i.e. the best fitted curve by the least squares methods minimizes the sum of squared difference between an observed value and the fitted value provided by a model.

Suppose that the data points are $(r_1, f_{r_1}), (r_2, f_{r_2}), (r_3, f_{r_3}), ..., (r_m, f_{r_m}),$ where $r_i$ is the independent variable, and $f_{r_i}$ is the dependent variable. The fitting curve has the deviation (error) from each data point, i.e. $d_1 = f_{r_1} - r_1,\ d_2 = f_{r_2} - r_2,\ d_3 = f_{r_3} - r_3, \cdots,\ d_m = f_{r_m} - r_m$ .

According to the techniques of least squares, the best fit curve has the property that $R=\sum_{r=1}^{m} d_i^2$, is minimum.

The $n-$ point scheme means the scheme takes $n$ consecutive points from the initial polygon to compute new point in order to get a refined polygon. Here $n$ is called the complexity of the scheme. For example, if the scheme has complexity 10 then it means the scheme uses 10 consecutive points from the initial polygon to compute new point to get refined polygon. Mustafa et al. (2015) pointed out that the less complexity, and the very complexity schemes are not suitable to handle noisy data. Therefore, in this paper, a 10-point approximating subdivision scheme is constructed by fitting a polynomial of degree 9 with the help of least squares technique. The proposed strategy can also be used to generate lower and higher complexity schemes. It can also be used to produce ternary, quaternary, $\cdots$, $b$-ary schemes $b \geqslant 3$.

The paper is organized as follows: In Section 2, a 10-point scheme is introduced. In Section 3, we discuss continuity, polynomial generation, polynomial reproduction, local analysis of the scheme. Applications of the scheme are presented in Section 4.

## 2. A 10-point scheme

Consider the following polynomial of degree 9 to determine the best curve fit to data/observations based on least squares technique

$$f(r) = f_r = \beta_0 + \beta_1 r + \beta_2 r^2 + \beta_3 r^3 + \beta_4 r^4 + \beta_5 r^5 + \beta_6 r^6 + \beta_7 r^7 + \beta_8 r^8 + \beta_9 r^9, \tag{2}$$

with respect to the observations $(x_r = r, f_r)$ for $r = -n+1, \ldots, n,$ where $n \geqslant 5.$ We want to determine the values of unknown parameters (betas) in (2) to make the following sum of squares of residuals $R$ a minimum:

$$R = \sum_{r=-n+1}^{n} [f_r - (\beta_0 + \beta_1 r + \beta_2 r^2 + \beta_3 r^3 + \beta_4 r^4 + \beta_5 r^5 + \beta_6 r^6 + \beta_7 r^7 + \beta_8 r^8 + \beta_9 r^9)]^2 .$$

Differentiating $R$ with respect to $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$ & $\beta_9$, and setting them to zero lead to a system of 10 normal linear equations. By solving it we get the following values of unknowns.

$$\beta_0 = \left(\frac{315}{64}\right)\left(\frac{1}{\alpha_0}\right)\gamma_0,$$

$$\alpha_0 = \frac{1}{n(64n^6 - 560n^4 + 1036n^2 - 225)(16n^4 - 520n^2 + 3969)},$$

$$\gamma_0 = \sum_{r=-n+1}^{n} (\overset{1}{\chi}_r + \overset{2}{\chi}_r) f_r,$$

$$\overset{1}{\chi}_r = 630n^{10} - 9240n^8r^2 + 36036n^6r^4 - 51480n^4r^6 + 24310n^2r^8 465n^8r + 60060n^6r^3$$
$$- 270270n^4r^5 + 437580n^2r^7 - 230945r^9 - 21945n^8 + 198660n^6r^2 - 270270n^4r^4$$
$$- 368940n^2r^6 + 546975r^8 + 127050n^6r - 1651650n^4r^3 + 4954950n^2r^5 - 4011150r^7$$

$$\overset{2}{\chi}_r = 222684n^6 - 515130n^4r^2 - 3939936n^2r^4 + 6756750r^6 - 1435665n^4r + 12222210n^2r^3$$
$$- 18002985r^5 - 621885n^4 - 5304970n^2r^2 + 19594575r^4 + 5517600n^2r - 22336600r^3$$
$$- 264924n^2 + 13018500r^2 - 5314320r + 907200,$$

$$\beta_1 = -\left(\frac{165}{128}\right)\left(\frac{1}{\alpha_1}\right)\gamma_1,$$

$$\alpha_1 = (4n^4 - 5n^2 + 1)(16n^6 - 200n^4 + 769n^2 - 900)$$
$$\times(16n^8 - 920n^6 + 19273n^4 - 174105n^2 + 571536)n,$$

$$\gamma_1 = \sum_{r=-n+1}^{n} (\overset{3}{\chi}_r + \overset{4}{\chi}_r + \overset{5}{\chi}_r) f_r,$$

$$\overset{3}{\chi}_r = 2940n^{16} - 77616n^{14}r^2 + 360360n^{12}r^4 - 560560n^{10}r^6 + 278460n^8r^8 - 48510n^{14}r$$
$$+ 840840n^{12}r^3 - 3783780n^{10}r^5 + 6126120n^8r^7 - 3233230n^6r^9 - 151410n^{14}$$
$$+ 2910600n^{12}r^2 - 8228220n^{10}r^4 + 4204200n^8r^6 + 2366910n^6r^8 + 2870175n^{12}r$$
$$- 42042000n^{10}r^3 + 504350n^8r^5 - 193993800n^6r^7 + 72747675n^4r^9 + 2862825n^{12}$$
$$- 30457812n^{10}r^2$$

$$\overset{4}{\chi}_r = -22612590n^8r^4 + 213993780n^6r^6 - 154577787n^4r^8 - 66596145n^{10}r$$
$$+ 803632830n^8r^3 - 2317985670n^6r^5 + 2081349270n^4r^7 - 431636205n^2r^9$$
$$- 22896335n^{10} - 73423350n^8r^2 + 1874142270n^6r^4 - 3292679390n^4r^6$$
$$+ 1068248805n^2r^8 + 776050275n^8r - 7407600200n^6r^3 + 15642551925n^4r^5$$
$$- 8647674750n^2r^7 + 607385350r^9 + 43491105n^8 + 3149091792n^6r^2,$$

$$\chi_r^5 = -16458034593n^4r^4 + 15182056890n^2r^6 - 1547525538r^8 - 4829843865n^6r + 34223178990n^4r^3 - 46679097465n^2r^5 + 10549324500r^7 + 414265425n^6 - 19425143430n^4r^2 + 52986118185n^2r^4 - 19074034980r^6 + 15762204150n^4r - 73891517700n^2r^3 + 47347850550r^5 - 2420539870n^4 + 45106289976n^2r^2 - 55081188162r^4 - 24443752080n^2r + 58745258000r^3 + 4368901320n^2 - 36215249400r^2 + 13976661600r - 2385936000,$$

$$\beta_2 = -\left(\frac{495}{32}\right)\left(\frac{1}{\alpha_2}\right)\gamma_2,$$

$$\alpha_2 = (4n^4 - 5n^2 + 1)(16n^6 - 200n^4 + 769n^2 - 900) \times (16n^8 - 920n^6 + 19273n^4 - 174105n^2 + 571536)n$$

$$\gamma_2 = \sum_{r=-n+1}^{n} (\chi_r^6 + \chi_r^7 + \chi_r^8) f_r,$$

$$\chi_r^6 = 2940n^{16} - 77616n^{14}r^2 + 360360n^{12}r^4 - 560560n^{10}r^6 + 278460n^8r^8 - 48510n^{14}r + 840840n^{12}r^3 - 3783780n^{10}r^5 + 6126120n^8r^7 - 3233230n^6r^9 - 151410n^{14} + 2910600n^{12}r^2 - 8228220n^{10}r^4 + 4204200n^8r^6 + 2366910n^6r^8 + 2870175n^{12}r - 42042000n^{10}r^3 + 154504350n^8r^5 - 193993800n^6r^7 + 72747675n^4r^9 + 2862825n^{12} - 30457812n^{10}r^2 - 22612590n^8r^4,$$

$$\chi_r^7 = 213993780n^6r^6 - 154577787n^4r^8 - 66596145n^{10}r + 803632830n^8r^3 - 2317985670n^6r^5 + 2081349270n^4r^7 - 431636205n^2r^9 - 22896335n^{10} - 73423350n^8r^2 + 1874142270n^6r^4 - 3292679390n^4r^6 + 1068248805n^2r^8 + 776050275n^8r - 7407600200n^6r^3 + 15642551925n^4r^5 - 8647674750n^2r^7 + 607385350r^9 + 43491105n^8 + 3149091792n^6r^2 - 16458034593n^4r^4,$$

$$\chi_r^8 = 15182056890n^2r^6 - 1547525538r^8 - 4829843865n^6r + 34223178990n^4r^3 - 46679097465n^2r^5 + 10549324500r^7 + 414265425n^6 - 19425143430n^4r^2 + 52986118185n^2r^4 - 19074034980r^6 + 15762204150n^4r - 73891517700n^2r^3 + 47347850550r^5 - 2420539870n^4 + 45106289976n^2r^2 - 55081188162r^4 - 24443752080n^2r + 58745258000r^3 + 4368901320n^2 - 36215249400r^2 + 13976661600r - 2385936000,$$

$$\beta_3 = \left(\frac{715}{32}\right)\left(\frac{1}{\alpha_3}\right)\gamma_3,$$

$$\alpha_3 = (6n^8 - 120n^6 + 273n^4 - 205n^2 + 36)(4n^2 - 25)\, n(n^2 - 9)(4n^4 - 113n^2 + 784)(4n^2 - 81),$$

$$\gamma_3 = \sum_{r=-n+1}^{n} (\chi_r^{9} + \chi_r^{10} + \chi_r^{11}) f_r,$$

$$\begin{aligned}\chi_r^{9} =& -58212n^{14}r + 720720n^{12}r^3 - 2522520n^{10}r^5 + 3341520n^8r^7 - 1492260n^6r^9 \\ &+ 13230n^{14} - 582120n^{12}r^2 + 3783780n^{10}r^4 - 7567560n^8r^6 + 4594590n^6r^8 + 3395700n^{12}r \\ &- 37477440n^{10}r^3 + 113513400n^8r^5 - 124750080n^6r^7 + 43275540n^4r^9 - 760725n^{12} \\ &+ 29106000n^{10}r^2 - 160810650n^8r^4 + 264864600n^6r^6 - 126351225n^4r^8 - 77295834n^{10},\end{aligned}$$

$$\begin{aligned}\chi_r^{10} =& \, 736756020n^8r^3 - 1837655820n^6r^5 + 1532810916n^4r^7 - 335012370n^2r^9 \\ &+ 17218845n^{10} - 556361190n^8r^2 + 2495402910n^6r^4 - 3100807710n^4r^6 \\ &+ 934999065n^2r^8 + 884164050n^8r - 6990743760n^6r^3 + 13405031640n^4r^5 \\ &- 7360016040n^2r^7 + 637372670r^9 - 195862275n^8 + 5128338600n^6r^2 \\ &- 17483541075n^4r^4 + 14364400050n^2r^6 - 1708749900r^8 - 5426810466n^6r,\end{aligned}$$

$$\begin{aligned}\chi_r^{11} =& \, 33445936524n^4r^3 - 43647433830n^2r^5 + 11145605484r^7 + 1188565245n^6 \\ &- 23692970070n^4r^2 + 54603053505n^2r^4 - 21108087000r^6 + 17644803330n^4r \\ &- 75975179280n^2r^3 + 50588087550r^5 - 3760495200n^4 + 51155666100n^2r^2 \\ &- 61213452300r^4 - 27660129288n^2r + 64101510616r^3 + 5585413680n^2 \\ &- 40669794000r^2 + 16035117120r - 2834092800,\end{aligned}$$

$$\beta_4 = \left(\frac{45045}{64}\right)\left(\frac{1}{\alpha_4}\right)\gamma_4,$$

$$\begin{aligned}\alpha_4 =& (256n^{14} - 8960n^{12} + 119392n^{10} - 766480n^8 + 2475473n^6 \\ &- 3822910n^4 + 2400129n^2 - 396900)n(4n^4 - 145n^2 + 1296),\end{aligned}$$

$$\beta_5 = -\left(\frac{15015}{64}\right)\left(\frac{1}{\alpha_5}\right)\sum_{r=-n+1}^{n} (\chi_r^{12} + \chi_r^{13}) f_r,$$

$$\begin{aligned}\chi_r^{12} &= -17820n^{12}r + 240240n^{10}r^3 - 884520n^8r^5 + 1211760n^6r^7 - 554268n^4r^9 \\ &\quad + 5670n^{12} - 249480n^{10}r^2 + 1621620n^8r^4 - 3243240n^6r^6 + 1969110n^4r^8 \\ &\quad + 926640n^{10}r - 10810800n^8r^3 + 33022080n^6r^5 - 35141040n^4r^7 + 11085360n^2r^9 \\ &\quad - 274050n^{10} + 10187100n^8r^2 - 54054000n^6r^4 + 83783700n^4r^6 - 36100350n^2r^8 \\ &\quad - 18102150n^8r + 175014840n^6r^3 - 414662976n^4r^5 + 300920400n^2r^7 - 43325282r^9,\end{aligned}$$

$$\begin{aligned}\chi_r^{13} &= 5044095n^8 - 152834220n^6r^2 + 624503880n^4r^4 - 661981320n^2r^6 + 131164605r^8 \\ &\quad + 169196280n^6r - 1276669680n^4r^3 + 2099719440n^2r^5 - 777029880r^7 - 44358300n^6 \\ &\quad + 1031377050n^4r^2 - 2913510600n^2r^4 + 1620268650r^6 - 786456066n^4r + 4156898460n^2r^3 \\ &\quad - 3647492394r^5 + 191612925n^4 - 3077742690n^2r^2 + 4698779085r^4 + 1685913900n^2r \\ &\quad - 4817913100r^3 - 369576900n^2 + 3121836300r^2 - 1230864624r + 217546560,\end{aligned}$$

$$\begin{aligned}\alpha_5 &= (64n^{12} - 1456n^{10} + 12012n^8 - 44473n^6 + 74074n^4 \\ &\quad - 48321n^2 + 8100)(4n^4 - 113n^2 + 784)(4n^2 - 81)n,\end{aligned}$$

$$\beta_6 = -\left(\frac{45045}{32}\right)\left(\frac{1}{\alpha_6}\right)\sum_{r=-n+1}^{n}(\chi_r^{14} + \chi_r^{15})f_r,$$

$$\begin{aligned}\chi_r^{14} &= 180n^{12} - 6160n^{10}r^2 + 32760n^8r^4 - 55440n^6r^6 + 29172n^4r^8 - 6930n^{10}r + 120120n^8r^3 \\ &\quad - 540540n^6r^5 + 875160n^4r^7 - 461890n^2r^9 - 4110n^{10} + 46200n^8r^2 + 278460n^6r^4 \\ &\quad - 1395240n^4r^6 + 1239810n^2r^8 + 306075n^8r - 4204200n^6r^3 + 13963950n^4r^5 \\ &\quad - 14586000n^2r^7 + 3464175r^9 - 13185n^8 + 2351580n^6r^2 - 17645082n^4r^4 \\ &\quad + 29655780n^2r^6,\end{aligned}$$

$$\begin{aligned}\chi_r^{15} &= -10567557r^8 - 4777080n^6r + 49219170n^4r^3 - 110330220n^2r^5 + 60167250r^7 \\ &\quad + 913320n^6 - 36183290n^4r^2 + 149292780n^2r^4 - 124552890r^6 + 32570175n^4r \\ &\quad - 228006350n^2r^3 + 270044775r^5 - 7403745n^4 + 166835790n^2r^2 - 338940693r^4 \\ &\quad - 93392640n^2r + 335049000r^3 + 20115540n^2 - 209604780r^2 + 79714800r \\ &\quad - 13608000,\end{aligned}$$

$$\begin{aligned}\alpha_6 &= (16n^6 - 56n^4 + 49n^2 - 9)(4n^4 - 41n^2 + 100) \\ &\quad \times (16n^8 - 920n^6 + 19273n^4 - 174105n^2 + 571536)n,\end{aligned}$$

$$\beta_7 = -\left(\frac{36465}{32}\right)\left(\frac{1}{\alpha_7}\right)\sum_{r=-n+1}^{n}(\chi_r^{16} + \chi_r^{17}) f_r,$$

$$\chi_r^{16} = -4620n^{10}r + 65520n^8r^3 - 249480n^6r^5 + 350064n^4r^7 - 163020n^2r^9 + 1890n^{10}$$
$$-83160n^8r^2 + 540540n^6r^4 - 1081080n^4r^6 + 656370n^2r^8 + 207900n^8r - 2446080n^6r^3$$
$$+7234920n^4r^5 - 7001280n^2r^7 + 1684540r^9 - 74025n^8 + 2633400n^6r^2 - 13063050n^4r^4$$
$$+18018000n^2r^6 - 6016725r^8 - 3289440n^6r$$

$$\chi_r^{17} = +30055116n^4r^3 - 61954200n^2r^5 + 32176716r^7 + 1035720n^6 - 28253610n^4r^2$$
$$+97837740n^2r^4 - 74324250r^6 + 22649220n^4r - 144314040n^2r^3 + 159976740r^5$$
$$-6279525n^4 + 117389250n^2r^2 - 215540325r^4 - 65491140n^2r + 218541284r^3$$
$$+15295140n^2 - 143203500r^2 + 56461680r - 9979200,$$

$$\alpha_7 = (256n^{16} - 13056n^{14} + 262752n^{12} - 2676752n^{10} + 14739153n^8$$
$$-43430478n^6 + 63566689n^4 - 38798964n^2 + 6350400)(4n^2 - 81)n,$$

$$\beta_8 = \left(\frac{109395}{64}\right)\left(\frac{1}{\alpha_8}\right)\sum_{r=-n+1}^{n}(\chi_r^{18} + \chi_r^{19}) f_r,$$

$$\chi_r^{18} = n^{10} - 2520n^8r^2 + 13860n^6r^4 - 24024n^4r^6 + 12870n^2r^8 - 3465n^8r + 60060n^6r^3$$
$$-270270n^4r^5 + 437580n^2r^7 - 230945r^9 - 525n^8 - 21420n^6r^2 + 348810n^4r^4$$
$$-1021020n^2r^6 + 778635r^8 + 127050n^6r - 1651650n^4r^3 + 4954950n^2r^5,$$

$$\chi_r^{19} = -4011150r^7 - 28140n^6 + 1398894n^4r^2 - 7549080n^2r^4 + 8702694r^6 - 1435665n^4r$$
$$+12222210n^2r^3 - 18002985r^5 + 372575n^4 - 9667410n^2r^2 + 22962555r^4 + 5517600n^2r$$
$$-22336600r^3 - 1251180n^2 + 14004756r^2 - 5314320r + 907200,$$

$$\alpha_8 = (1024n^{18} - 72960n^{16} + 2108544n^{14} - 31989920n^{12} + 275773524n^{10} - 1367593305n^8$$
$$+3772135474n^6 - 5304097665n^4 + 3168117684n^2 - 514382400)n,$$

$$\beta_9 = -\left(\frac{230945}{64}\right)\left(\frac{1}{\alpha_9}\right)\sum_{r=-n+1}^{n}(-630n^8r + 9240n^6r^3 - 36036n^4r^5 + 51480n^2r^7 - 24310r^9$$
$$+315n^8 - 13860n^6r^2 + 90090n^4r^4 - 180180n^2r^6 + 109395r^8 + 23520n^6r - 267960n^4r^3$$
$$+720720n^2r^5 - 531960r^7 - 9450n^6 + 311850n^4r^2 - 1351350n^2r^4 + 1351350r^6$$
$$-269934n^4r + 2074380n^2r^3 - 2816814r^5 + 85995n^4 - 1850310n^2r^2 + 3918915r^4$$
$$+1051860n^2r - 3946580r^3 - 258300n^2 + 2603700r^2 - 1026576r + 181440)\, f_r,$$

$$\alpha_9 = (4n^4 - 113n^2 + 784)(n^2-1)(n^2-9)$$
$$\times(64n^8 - 816n^6 + 3276n^4 - 4369n^2 + 900)(4n^2-81)n.$$

By substituting all $\beta$'s in (2) and taking $n = 5$, $r = 1/4$ and $r = 3/4$, we get

$$f(1/4) = \frac{13585}{33554432}f_{-4} - \frac{159885}{33554432}f_{-3} + \frac{230945}{8388608}f_{-2} - \frac{969969}{8388608}f_{-1} + \frac{14549535}{16777216}f_0$$
$$+\frac{4849845}{16777216}f_1 - \frac{692835}{8388608}f_2 + \frac{188955}{8388608}f_3 - \frac{138567}{33554432}f_4 + \frac{12155}{33554432}f_5,$$

$$f(3/4) = \frac{12155}{33554432}f_{-4} - \frac{138567}{33554432}f_{-3} + \frac{188955}{8388608}f_{-2} - \frac{692835}{8388608}f_{-1} + \frac{4849845}{16777216}f_0$$
$$+\frac{14549535}{16777216}f_1 - \frac{969969}{8388608}f_2 + \frac{230945}{8388608}f_3 - \frac{159885}{33554432}f_4 + \frac{13585}{33554432}f_5.$$

If $k$ represents subdivision level and $i \in \mathbb{Z}$, then from the above rules we get the following 10-point approximating scheme.

$$f_{2i}^{k+1} = \left(\frac{1}{\zeta}\right)\{13585 f_{i-4}^k - 159885 f_{i-3}^k + 923780 f_{i-2}^k - 3879876 f_{i-1}^k + 29099070 f_i^k$$
$$+9699690 f_{i+1}^k - 2771340 f_{i+2}^k + 755820 f_{i+3}^k - 138567 f_{i+4}^k + 12155 f_{i+5}^k\},$$

$$f_{2i+1}^{k+1} = \left(\frac{1}{\zeta}\right)\{12155 f_{i-4}^k - 138567 f_{i-3}^k + 755820 f_{i-2}^k - 2771340 f_{i-1}^k + 9699690 f_i^k$$
$$+29099070 f_{i+1}^k - 3879876 f_{i+2}^k + 923780 f_{i+3}^k - 159885 f_{i+4}^k + 13585 f_{i+5}^k\}, \tag{3}$$

where $\zeta = 33554432$, $f_i^{k+1}$ and $f_i^k$ are control points at refine and coarse level of iterations, respectively.

**Remark 1:**

We get different complexity schemes at different values of $n$. For example, at $n=6$ and 7, we get 12- and 14-point approximating schemes, respectively. By taking $n=5$ and evaluating (2) at $r=1/6$, $r=3/6$ and $r=5/6$, we get 10-point ternary approximating scheme. Similarly, we get even-point quaternary, quinary and higher arity approximating schemes.

## 3. Analysis of the scheme

In this section, we present the analysis of 10-point scheme by using the methodology presented in Hormann (2012). Analysis includes continuity, reproduction, generation and limit analysis of the scheme.

### 3.1.Continuity

Since the sum of coefficients in the refinement rule of $f_{2i}^{k+1}$ and the sum of coefficients in the refinement rule of $f_{2i+1}^{k+1}$ are both equal to one, the necessary condition for the convergence of the scheme is satisfied. The Laurent polynomial of the scheme is

$$\begin{aligned}\alpha(z)=&\left(\frac{1}{33554432z^{10}}\right)[13585z^0+12155z^1-159885z^2-138567z^3+923780z^4\\&+755820z^5-3879876z^6-2771340z^7+29099070z^8+9699690z^9+9699690z^{10}\\&+29099070z^{11}-2771340z^{12}-3879876z^{13}+755820z^{14}+923780z^{15}-138567z^{16}\\&-159885z^{17}+12155z^{18}+13585z^{19}].\end{aligned} \tag{4}$$

It can be written as

$$\alpha(z)=\left(\frac{1+z}{2}\right)b(z),$$

where

$$\begin{aligned}b(z)=&\left(\frac{1}{16777216z^{10}}\right)[12155z^0+1430z^1-139997z^2-19888z^3+775708z^4+148072z^5\\&-2919412z^6-960464z^7+10660154z^8+18438916z^9+10660154z^{10}-960464z^{11}\\&-2919412z^{12}+148072z^{13}+775708z^{14}-19888z^{15}-139997z^{16}+1430z^{17}+12155z^{18}].\end{aligned}$$

Let $S_c$ be the difference scheme corresponding to $c(z)$ obtained from $b(z)$, where

$$c(z)=\left(\frac{1}{16777216z^{10}}\right)[12155z^{0}-10725z^{1}-129272z^{2}+109384z^{3}+666324z^{4}$$
$$-518252z^{5}-2401160z^{6}+1440696z^{7}+9219458z^{8}+9219458z^{9}+1440696z^{10}$$
$$+1440696z^{11}-2401160z^{12}-518252z^{13}+666324z^{14}+109384z^{15}-129272z^{16}$$
$$-10725z^{17}+12155z^{18}],$$

then $\| S_c \|_\infty =$ max[ sum of modulus of odd coefficients in $c(z)$, sum of modulus of even coefficients in $c(z)$] = 14507426/16777216 = 0.8647 $<$ 1. So $S_c$ is contractive and the scheme $S_b$ corresponding to $b(z)$ is convergent. So, scheme $S_\alpha$ i.e. scheme (3) is $C^1$ - continuous.

Now we write $\alpha(z)$ as follows

$$\alpha(z)=\left(\frac{1+z}{2}\right)^{2} b_1(z),$$

where

$$b_1(z)=\left(\frac{1}{8388608z^{10}}\right)[12155z^{0}-10725z^{1}-129272z^{2}+109384z^{3}+666324z^{4}$$
$$-518252z^{5}-2401160z^{6}+1440696z^{7}+9219458z^{8}+9219458z^{9}+1440696z^{10}$$
$$-2401160z^{11}-518252z^{12}+666324z^{13}+109384z^{14}-129272z^{15}-10725z^{16}$$
$$+12155z^{17}].$$

Again let $S_{c_1}$ be the difference scheme corresponding to $c_1(z)$ obtained from $b_1(z)$, where

$$c_1(z)=\left(\frac{1}{8388608z^{10}}\right)[12155z^{0}-35035z^{1}-71357z^{2}+287133z^{3}+163415z^{4}$$
$$-1132215z^{5}-300145z^{6}+3173201z^{7}+3173201z^{8}-300145z^{9}-1132215z^{10}$$
$$+163415z^{11}+287133z^{12}-71357z^{13}-35035z^{14}+12155z^{15}],$$

then $\| S_{c_1} \|_\infty = 0.6168 < 1$. So scheme $S_{c_1}$ is contractive and $S_{b_1}$ is convergent. So, $S_\alpha$ is $C^2-$ continuous.

Again $\alpha(z)$ can be written as

$$\alpha(z)=\left(\frac{1+z}{2}\right)^{3} b_2(z), (5)$$

$$b_2(z)=\left(\frac{1}{4194304z^{10}}\right)[12155z^0-22880z^1-106392z^2+215776z^3+450548z^4$$
$$-968800z^5-1432360z^6+2873056z^7+634602z^8+2873056z^9-1432360z^{10}$$
$$-968800z^{11}+450548z^{12}+215776z^{13}-106392z^{14}-22880z^{15}+12155z^{16}].$$

Let $S_{c_2}$ be the difference scheme corresponding to $c_2(z)$ obtained from $b_2(z)$, where

$$c_2(z)=\left(\frac{1}{4194304z^{10}}\right)[12155z^0-47190z^1-24167z^2+311300z^3-147885z^4$$
$$-984330z^5+684185z^6+2489016z^7+684185z^8-984330z^9-147885z^{10}$$
$$+311300z^{11}-24167z^{12}-47190z^{13}+12155z^{14}],$$

then $\| S_{c_2} \|_\infty = 1.233 > 1.$

So $S_{c_2}$ is not contractive and $S_{b_2}$ is not convergent. So, $S_\alpha$ is not $C^3-$continuous. Thus, proposed 10-point scheme is $C^2$-continuous.

### 3.2. Polynomial generation and reproduction

In this section, we will discuss the degree of the polynomial generation and polynomial reproduction of the proposed scheme.

Since by (5)

$$\alpha(z)=\left(\frac{1+z}{2}\right)^{2+1} b_2(z),$$

then the degree of polynomial generation is 2.

Now by taking derivative of $\alpha(z)$ with respect to $z$ and then by substituting $z=1$, we get $\alpha'(1)=-1$. This implies that $\tau=\frac{\alpha'(1)}{2}=-\frac{1}{2}$. This further implies that

$$t_i^k=-\tau+\frac{i+\tau}{2^k}=\frac{1}{2}+\frac{i-1/2}{2^k}.$$

Thus, by Hormann (2012) the proposed scheme has dual parameterization. Since

$$\alpha^k(1)=2\prod_{l=0}^{k-1}\left(-\frac{1}{2}-l\right) \text{ and } \alpha^k(-1)=0,$$

for $k = 0,1,2,$ where $\alpha^k(z)$ is the $k$ derivative of $\alpha(z)$ then, again by Hormann (2012), a 10-point binary subdivision scheme reproduces polynomials of degree 2 with respect to the parameterization

$$t_i^k = \frac{1}{2} + \frac{i-1/2}{2^k}.$$

### 3.3. Local Analysis

In this section, by using local analysis, we can find the limit position of control point of the initial control polygon on the limit curve with the help of limit stencil. For this, we consider the following matrix representation of scheme (3) for $i = -4,-3,-2,-1,0,1,2,3,4$,

$$F^{k+1} = SF^k,$$

$$F^{k+1} = [f_{-8}^{k+1}, f_{-7}^{k+1}, f_{-6}^{k+1}, f_{-5}^{k+1}, f_{-4}^{k+1}, f_{-3}^{k+1}, f_{-2}^{k+1}, f_{-1}^{k+1}, f_{0}^{k+1}, f_{1}^{k+1}, f_{2}^{k+1}, f_{3}^{k+1}, f_{4}^{k+1}, f_{5}^{k+1}, f_{6}^{k+1}, f_{7}^{k+1}, f_{8}^{k+1}, f_{9}^{k+1}]^T,$$

$$F^{k} = [f_{-8}^{k}, f_{-7}^{k}, f_{-6}^{k}, f_{-5}^{k}, f_{-4}^{k}, f_{-3}^{k}, f_{-2}^{k}, f_{-1}^{k}, f_{0}^{k}, f_{1}^{k}, f_{2}^{k}, f_{3}^{k}, f_{4}^{k}, f_{5}^{k}, f_{6}^{k}, f_{7}^{k}, f_{8}^{k}, f_{9}^{k}]^T,$$

$$S = \frac{1}{\zeta}\begin{pmatrix}
\tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{t} & \tilde{s} & \tilde{r} & \tilde{q} & \tilde{p} & \tilde{o} & \tilde{n} & \tilde{m} & \tilde{l} & \tilde{\kappa} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\kappa} & \tilde{l} & \tilde{m} & \tilde{n} & \tilde{o} & \tilde{p} & \tilde{q} & \tilde{r} & \tilde{s} & \tilde{t}
\end{pmatrix},$$

where $\zeta = 33554432$, $\tilde{\kappa} = 13585$, $\tilde{l} = -159885$, $\tilde{m} = 923780$, $\tilde{n} = -3879876$, $\tilde{o} = 29099070$, $\tilde{p} = 9699690$, $\tilde{q} = -2771340$, $\tilde{r} = 755820$, $\tilde{s} = -138567$ and $\tilde{t} = 12155$.

The invariant neighborhood size of the matrix $S$ is 18 and its eigenvalues are:

$$\lambda = 1.000,\ 0.5000,\ 0.2500,\ 0.1250,\ 0.0625, -0.0287,\ 0.0346,\ 0.0312,\ -0.0198,$$
$$0.0156,\ 0.0104,\ 0.0089,\ 0.0078,\ -0.0026,\ -0.0025,\ 0.0039,\ 0.0020,\ 0.0010.$$

If $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9, \eta_{10}, \eta_{11}, \eta_{12}, \eta_{13}, \eta_{14}, \eta_{15}, \eta_{16}, \eta_{17}$ and $\eta_{18}$ are the eigenvectors corresponding to these eigenvalues, then $Q$ is the matrix whose columns are these eigenvectors.
If $D$ is a diagonal matrix whose diagonal entries are the above eigenvalues, then by the eigenvalue decomposition $S = QDQ^{-1}$. By performing simple algebraic operation we get $S^k = QD^kQ^{-1}$. Since the largest eigenvalue is one and other values are less than one so $\lim_{k\to\infty} D^k$ is a matrix whose first left top entry is one and other entries are zeros. Since $f^{k+1} = Sf^k = S(Sf^{k-1}) = S^2 f^{k-2} = .... = S^k f^0$ then $f^{k+1} = (QD^kQ^{-1}) f^0$. Taking limit, we get $f^\infty = Q(\lim_{k\to\infty} D^k)Q^{-1} f^0$. This implies that

$$\begin{pmatrix} f_{-8}^\infty \\ f_{-7}^\infty \\ f_{-6}^\infty \\ f_{-5}^\infty \\ f_{-4}^\infty \\ f_{-3}^\infty \\ f_{-2}^\infty \\ f_{-1}^\infty \\ f_{0}^\infty \\ f_{1}^\infty \\ f_{2}^\infty \\ f_{3}^\infty \\ f_{4}^\infty \\ f_{5}^\infty \\ f_{6}^\infty \\ f_{7}^\infty \\ f_{8}^\infty \\ f_{9}^\infty \end{pmatrix} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \\ \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \end{pmatrix} \begin{pmatrix} f_{-8}^0 \\ f_{-7}^0 \\ f_{-6}^0 \\ f_{-5}^0 \\ f_{-4}^0 \\ f_{-3}^0 \\ f_{-2}^0 \\ f_{-1}^0 \\ f_{0}^0 \\ f_{1}^0 \\ f_{2}^0 \\ f_{3}^0 \\ f_{4}^0 \\ f_{5}^0 \\ f_{6}^0 \\ f_{7}^0 \\ f_{8}^0 \\ f_{9}^0 \end{pmatrix}.$$

Hence, the limit stencil is

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} & \hat{l} & \hat{m} & \hat{n} & \hat{o} & \hat{p} & \hat{q} & \hat{r} & \hat{s} & \hat{t} & \hat{u} & \hat{v} & \hat{w} & \hat{x} & \hat{y} & \hat{z} \end{bmatrix},$$

where

$$\hat{i} = -0.0000,\ \hat{j} = -0.0000,\ \hat{k} = -0.0000,\ \hat{l} = -0.0000,\ \hat{m} = 0.0007,\ \hat{n} = -0.0070,$$
$$\hat{o} = 0.0365,\ \hat{p} = -0.1370,\ \hat{q} = 0.6068,\quad \hat{r} = 0.6068,\ \hat{s} = -0.1370,\ \hat{t} = 0.0365,$$
$$\hat{u} = -0.0070,\ \hat{v} = 0.0007,\ \hat{w} = -0.0000,\ \hat{x} = -0.0000,\ \hat{y} = -0.0000,\ \hat{z} = -0.0000.$$

This means that when we apply this stencil to the initial points

$$f_{-8}^0, f_{-7}^0, f_{-6}^0, f_{-5}^0, f_{-4}^0, f_{-3}^0, f_{-2}^0, f_{-1}^0, f_0^0, f_1^0, f_2^0, f_3^0, f_4^0, f_5^0, f_6^0, f_7^0, f_8^0, f_9^0$$

we get the limit position of the central one i.e., $f_0^0$.

## 4. Applications and conclusion

Here are the numerical examples (see Figures 1-2), which show that the proposed scheme is suitable for fitting data. Different types of data have been fitted by 10-point approximating schemes.

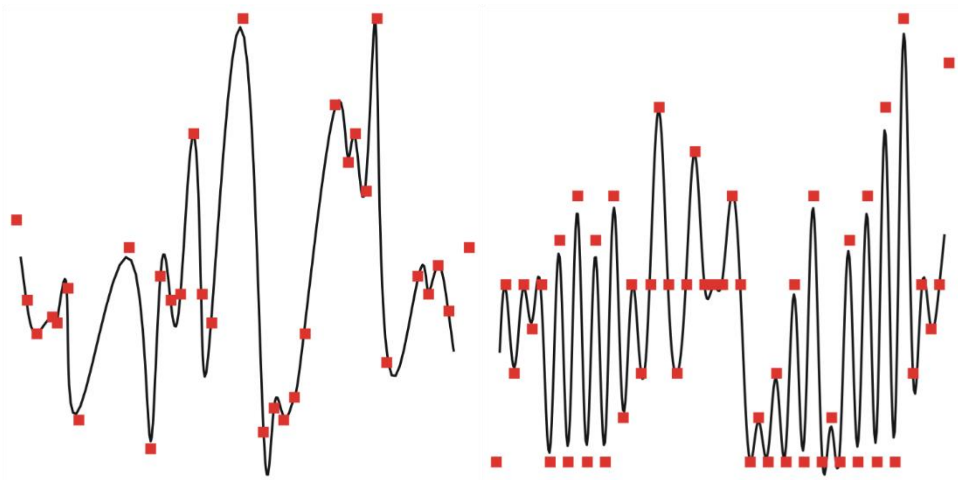

**Figure 1.** Limit curve is fitted by 10-point scheme whereas solid diamonds show the initial control points.

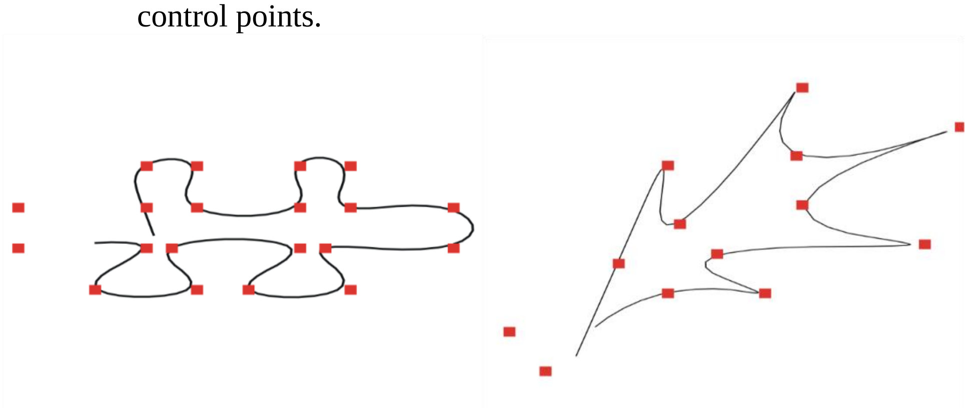

**Figure 2.** Limit curve is fitted by 10-point scheme whereas solid diamonds show the initial control points.

### 4.1. Conclusion

In this paper, a 10-point approximating scheme is presented by fitting 9-degree polynomial to data by least squares algorithm. Analysis of 10-point scheme is also presented. Analysis includes continuity, polynomial generation and reproduction and local limiting analysis. It is shown that proposed scheme is $C^2$-continuous that is fitting curve to data is smooth enough. The polynomial generation and reproduction of the scheme is 2. Limiting mask/stencil is also presented to find point on the limiting curve. Results show that 10-point scheme has nearly interpolating behavior. The families of $2m-$and $(2m+1)$-point binary, ternary, $\ldots, b$-ary approximating schemes by fitting different degree polynomials to data by least squares algorithm can be constructed by proposed framework.

## *Acknowledgement:*

*The authors would like to thank the referees for their helpful suggestions and comments which showed the way to improve this work. This work is supported by NRPU (P. No. 3183) HEC Pakistan.*

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