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Preferred Distribution Function for Estimating $P(Y < X)$

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Abstract: The estimation of $P(Y < X)$, assuming X and Y to be independently and identically distributed random variables, has been extensively studied in the literature. Distribution functions such as Burr Type X, Exponential, and Gamma have been considered. In this paper, we consider same type of distributions. Utilizing the Maximum Likelihood Estimation method and using simulation we will estimate the unknown parameters of these distributions. To choose a preferred distribution, comparison among results will be made based on the ratio of the parameters and the sample size.

Key Words: Reliability, Probability Distributions, MLE, Populations' Parameters, Samples' sizes, Simulation

1. INTRODUCTION AND BACKGROUND

The reliability of an item, or a product, is becoming the top priority for the third millennium and technically sophisticated customer. Manufacturers and all other producing entities are sharpening their tools to satisfy that customer. Estimation of that reliability has become a concern for many quality professionals and statisticians. When Y represents the random value of a stress that a device will be subjected to in service, and X represents the strength that varies from item to item in the population of devices, then the reliability R is P (Y < X), i.e., the probability that a randomly selected device functions successfully. Different distributions have been assumed for the random variables X and Y. Downton (1973), and Church and Harris (1970) have discussed the estimation of R = P (Y < X) in the normal and gamma cases respectively.

The estimation of P (Y < X) in the Burr type X model has been referred to in the literature by Awad and Gharraf (1986), who provided a simulation study which compares three estimates for R = P(Y < X). Those estimators are: the minimum variance unbiased, the maximum likelihood, and Bayes estimators. Also the sensitivity of the Bayes estimator to the prior parameters was considered. Ahmad, Fakhry & Jaheen (1997), deals with the estimation of R = P(Y < X) in the Burr type X case, when maximum likelihood, Bayes and empirical techniques were used. Monte-Carlo simulation was carried out to compare the three methods of estimation. In addition to the comparison among the three estimators, some characterization of the Burr type X distribution was presented. The first characteristic is based on the recurrence relationship between two successively conditional moments of a certain function of the random variable, whereas the second one was given by the conditional variance of that function. Surles & Padgett (1998) addressed the inference on R = P(Y < X). Some properties of Burr type X distribution were reviewed, and the existing and new results on estimation of R were discussed. A significance test for R was introduced based on the MLE of R. Also, Bayesian inference on R was presented when the parameters are assumed to have independent gamma distributions. An algorithm was given for finding the highest posterior density interval for R. The value of R = P (Y < X) has been calculated in the above cited references under the assumed distributions and using different methods of estimation. None of the references has addressed the effect of the sample size and the values of the distribution parameters on the estimation of R.

In this paper we check, through simulation, on the effect of the sample sizes, and parameters' values on estimating P (Y < X). The Burr type X model will be considered as the underlying assumption for the stress Y and strength X simultaneously in Section 2. Tables I, II, and III correspond to the cases when the ratio of the sample sizes m/n = 1, 2, and ½ respectively, and different ratios of the parameters' values are used to estimate R.

In section 3 Gamma distribution will be considered as the underlying assumption for the stress Y and strength X simultaneously. This distribution was chosen since it is a part of the parametric families of life distributions for which the failure rate is monotone. Tables IV, IV correspond to the cases of (M, N) = (1, 1), i.e., the exponential case, and (M, N) = (2, 2). Table VI has Right Tail Values of the F-Distribution with M=N, And the Corresponding Tail Area, with 2M and 2N degrees of freedom for the numerator and denominator respectively. Moreover, the gamma distribution is a good model for many nonnegative random variables of the continuous type. For instance, the distribution of certain incomes (and hence spending) could be modeled satisfactorily by the gamma distribution, since the two parameters in the distribution provide a great of flexibility, Hogg & Craig (1995).

Conclusions and recommendations are given in section 4

2. BURR TYPE X DISTRIBUTION

Surles and Padgett (1998) considered the inference on R = P(Y < X) when X and Y are independently distributed Burr type X random variables. New results of estimating R were also discussed. If X and Y are independent random variables whose distributions are Burr type X with parameters θ and λ respectively, Ahmad, Fakhry, and Jaheen (1997) have shown that R is given by

$$R = P(Y < X) = (1 + \lambda / \theta)^{-1} \qquad (1)$$

It is noting here that R increases as the ration λ / θ decreases, and thus the parameter λ of the stress Y ~ Burr X (λ) needs to be much smaller than θ of the strength X ~ Burr X (θ). The Maximum Likelihood Estimator (MLE) of R, in (1), is

$$\hat{R} = (1 + \hat{\lambda} / \hat{\theta})^{-1} \quad (2)$$

where $\hat{\lambda}$ and $\hat{\theta}$ are the MLE's of λ and θ respectively. They are expressed as

$$\hat{\theta} = m / \{-\sum_{i=1}^{n} \ln(1 - \exp(-x_i^2))\} \quad (3)$$

and

$$\hat{\lambda} = n / \{-\sum_{i=1}^{m} \ln(1 - \exp(-y_j^2))\} \quad (4)$$

based on random samples of m and n from Burr X(θ) and Burr X(λ) populations respectively. When X has the one-parameter Burr type X distribution, the cumulative distribution function is

$$F(x|\theta) = \{1 - \exp(-x^2)\}^{\theta} \qquad x > 0, \ \theta > 0. \quad (5)$$

It is customary to utilize the above cumulative distribution function in the simulation. For a given θ , we can choose 0 < a < 1 and put $\mathrm{a} = (1 - \exp(-x^2))^{\theta}$ i.e.

$$(1/\theta) \ln \mathrm{a} = \ln\left[1 - \exp(-x^2)\right]. \quad (6)$$

Thus by considering different values of a, and solving for x, we generate the random sample $\underline{X}$ = *(*$X_1, X_2, \ldots, X_m$*)*. Similarly, for a given λ , and for 0 < b< 1we have

$$(1/\lambda) \ln b = \ln\left[1 - \exp(-y^2)\right] , \quad (7)$$

Again by considering different values for b, we generate the random sample $\underline{Y}$ = ($Y_1, Y_2, \ldots ,$ Y_n). Since X and Y are positive random variables, the above relationships, in (6) and (7), between a & x and b & y are one-to-one. Thus utilizing (6) and (7) in (2) we have

$$\hat{R} = \left[1 + (n\lambda \sum_{i=1}^{m} \ln a_i) / (m\theta \sum_{j=1}^{n} \ln b_j)\right]^{-1} \quad (8)$$

where $0 \le a_i \le 1, i = 1, 2, \ldots, m$; $0 \le b_j \le 1, j = 1,2, \ldots, n$; and λ & θ are pre-assigned values. Based on pre-assigned values for the ratio λ / θ , simulation was done on 5000 runs. The estimated values of R, with their standard deviations, are summarized in Tables I, II, and III for three cases when the ratio of the sample sizes *m/n* = 1, 2, and ½ respectively. The sample sizes m and n can assume any values but were taken to be less than 10 for the practical application of the study, especially when time is a factor in collecting the data.

TABLE I

Estimated Average Values of R and Their Standard Deviations
When m/n = 1, with m = 2, 3, 4, 5, and 10

λ / θ	R	2	3	4	5	10
1	½	0.5145	0.5145	0.5158	0.5147	0.5133
		0.2093	0.1742	0.1585	0.1452	0.1058
½	2/3	0.6492	0.6617	0.6748	0.6695	0.6719
		0.1958	0.1561	0.1493	0.1293	0.0934
1/3	¾	0.7253	0.7379	0.7497	0.7470	0.7517
		0.1795	0.1366	0.1280	0.1117	0.0793
¼	4/5	0.7708	0.7852	0.7960	0.7945	0.8000
		0.1656	0.1206	0.1115	0.0975	0.0682
1/5	5/6	0.8025	0.8178	0.8276	0.8268	0.8325
		0.1540	0.1077	0.0987	0.0863	0.0596
1/10	10/11	0.8807	0.8957	0.9023	0.9027	0.9074
		0.1169	0.0698	0.0626	0.0545	0.0361

TABLE II

Estimated Average Values of R and Their Standard Deviations
When m/n = 2, with m = 4, 6, 8, and 10, and m(n) values

λ / θ	R	4(2)	6(3)	8(4)	10(5)
1	½	0.4437	0.4839	0.4622	0.4787
		0.1905	0.1329	0.1414	0.0528
½	2/3	0.5899	0.6411	0.6193	0.6457
		0.1919	0.1261	0.1356	0.0490
1/3	¾	0.6711	0.7229	0.7030	0.7314
		0.1820	0.1123	0.1218	0.0424
¼	4/5	0.7219	0.7737	0.7559	0.7836
		0.1715	0.0998	0.1090	0.0368
1/5	5/6	0.7586	0.8086	0.7924	0.8188
		0.1617	0.0894	0.0982	0.0323
1/10	10/11	0.8513	0.8913	0.8807	0.9000
		0.1268	0.0581	00649	0.0198

TABLE III
Estimated Average Values of R and Their Standard Deviations
When m/n = 1/2, with n = 4, 6, 8, and 10, and n (m) values

λ/θ	R	4(2)	6(3)	8(4)	10(5)
1	½	0.5563	0.5161	0.5378	0.4991
		0.1905	0.1329	0.1414	0.0528
½	2/3	0.6951	0.6708	0.6884	0.6601
		0.1650	0.1161	0.1228	0.0892
1/3	¾	0.7651	0.7492	0.7634	0.7419
		0.1422	0.0987	0.1044	0.0766
¼	4/5	0.8082	0.7971	0.8090	0.7917
		0.1244	0.0851	0.0901	0.0664
1/5	5/6	0.8376	0.8295	0.8396	0.8253
		0.1104	0.0745	0.0791	0.0584
1/10	10/11	0.9074	0.9050	0.9107	0.9032
		0.0706	0.0457	0.0488	0.0360

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3. GAMMA DISTRIBUTION

Constantine, Karson, and Tse (1986) considered the estimation of R = P (Y < X) when X ~ Γ (M, λ) and Y ~ Γ (N, μ) are independent with M and N known integer-valued shape parameters, and μ & λ are the unknown scale parameters. They discussed different representations of the Maximum Likelihood Estimator (MLE) of R, and calculated their Mean Square Errors (MSE). If $\rho = \mu/\lambda$, then P(Y < X) = P (U > ρ), where U = (X/ λ) / (y/ μ). The MLEs of μ and λ are $\hat{\mu} = \overline{Y}/N$ and $\hat{\lambda} = \overline{X}/M$, respectively. The invariance property of the MLE implies that the MLE of ρ is $\hat{\rho} = \hat{\mu}/\hat{\lambda}$. Constantine, Karson, and Tse (1986), has several representations of R, and one of them, eq. (2.3), is given by R = 1 - F_F ((N/M) ρ ; 2M, 2N), where F_F (. ; 2M, 2N) is the cumulative distribution function of an F random variable with 2M numerator and 2N denominator degrees of freedom. It is clear to say that the value of F_F (. ; 2M, 2N) should be very small for R to be a good measure. Thus the argument for the CDF, namely (N/M) ρ , must be small as well. For the special case of M=N, we find that ρ Must be small. This, in its turn, implies that the value of μ should be much smaller than that of λ. Motivated by Constantine, Karson, and Tse (1986) statement: "Figure 3 and Table II provide results that are useful for studying the problem of sample size allocation. We note first from figure 3 that an unwise allocation of sample sizes can yield an unnecessary large values of

MSE", we will check on the effect of the sample sizes (m and n) on the value of R. In this section, we use the Uniformly Minimum Variance Unbiased Estimator (UMVUE) of R derived in the 2-sample gamma model. Using their notation, where $X_T = \sum_{i=1}^{m} X_i$, and $Y_T = \sum_{j=1}^{n} Yj$,

and W = X_T/Y_T, we have, with *I= (m-1)*M-1,* and *J = (n-1)*N-1:*

$$\text{when } \frac{1}{w} > 1,\ \hat{R} = \frac{\Gamma(mM)\Gamma(nN)}{\Gamma(M)\Gamma(N)} \sum_{j=0}^{J} (-1)^j \frac{\Gamma(M+N+j)}{\Gamma(mM+N+j)\Gamma(nN-N-j)\Gamma(j+1)} \frac{w^{N+j}}{N+j}$$

$$\text{When } \frac{1}{w} \le 1,\ \hat{R} = 1 - \frac{\Gamma(mM)\Gamma(nN)}{\Gamma(M)\Gamma(N)} \sum_{i=0}^{I} (-1)^i \frac{\Gamma(M+N+i)}{\Gamma(nN+M+i)\Gamma(mM-M-i)\Gamma(i+1)} \frac{w^{-M-i}}{M+i}$$

The above expressions for R will be used to check the effect of the parameter values M and N, of the underlying distributions, and the sample sizes m and n, on the estimation of R.

Our results will be partially compared with those values in Table II of Constantine, Karson, and Tse (1986) for the case (M, N) = (1, 1), i.e., the special case of the Gamma distribution, which is the Exponential. When (M, N) = (1, 1), we see that I = m – 2, and J = n – 2, and for I and J to be non-negative integers, it is imperative to have both m and n $\geq$ 2. Thus the following cases will be considered:

1. If $(m, n) = (2, 2), then (I, J) = (0, 0)$, and

   $\hat{R} = w/2$ $\qquad w < 1$, or

   $\hat{R} = 1 - 1(2w)$ $\qquad w \ge 1$.

2. If $(m, n) = (3, 3), then\,(I, J) = (1, 1)$, and

   $\hat{R} = (2/3)w[1 - w/4]$ $\qquad w < 1$,or

   $\hat{R} = 1 - 2/(3w)[1 - 1/(4w)]$ $\qquad w \ge 1$

Clearly the above expressions for $\hat{R}$ indicate that when w < 1 then R is bounded from above by ½, while it is bounded from below by ½ when w $\geq$ 1.

3. If $(m, n) = (4, 4), then\,(I, J) = (2, 2)$, and

   $\hat{R} = w/6\left[1/2 - w/5 + w^2/30\right]$ $\qquad w < 1$, or

   $\hat{R} = 1 - 1/(6w)\left[1/2 - 1/(5w) + 1/(30w^2)\right]$ $\qquad w \ge 1$.

It is worth noting that, when w = 1, the bound imposed on $\hat{R}$, in 1. & 2. above, is no longer applicable in this case. Moreover, as m and n increase, the values of $\hat{R}$ in 3.will almost complement each other. Simulation results are shown in Table III. In the exponential case, i/e., if M=N=1, for the shape parameters, and when $m=n$, with $w<1$, $\hat{R}$ is expressed as a polynomial in w of degree m-1; and if $w \geq 1$, $\hat{R}$ is a polynomial in 1/w of degree m-1.

4. When m and n are not equal, and for a fixed n, with $w < 1$, the value of $\hat{R}$ decreases as it is apparent from the following expressions for n=3, and $J = 1$, we have

$$\hat{R} = 2w/m\{1-w/(m+1)\}.$$

The following cases for (m, n) = (2, 3), (3, 3), and (4, 3) are simulated in Table IV, based on the formulas for (m, n) = (2, 3) and (I, J) = (0, 1)

$$\hat{R} = w(1-w/3) \qquad w<1; \text{ and}$$

$$\hat{R} = 1-1/(3w) \qquad w>1.$$

5. When m and n are not equal, and for a fixed m, with w > 1, the value of $\hat{R}$ increases as it is shown in

$$\hat{R} = 1-2/(nw)\{1-1/(n+1)w)\}.$$

The following cases for (m, n) = (3, 2), and (3, 4) are simulated in Table IV, based on the formulas for (m, n) = (3, 2) and (I, J) = (1, 0)

$$\hat{R} = w/3 \qquad w<1; \text{ and}$$

$$\hat{R} = 1-1/w\{1-1/(3w)\} \qquad w>1.$$

Table IV Exponential case, M=N=1
Estimated Average Values of R and Their Standard Deviations (in Parentheses)

(M, N)	(m, n)	$\hat{R}$, W<1	$\hat{R}$, W≥1
(1, 1)	(2, 2)	0.2475 2470 (0.130)	0.7507 2530 (0.131)
	(2, 3)	0.3633 3439 (0.174)	0.8010 1561 (0.079)
	(2, 4)	0.4219 4024 (0.193)	0.8296 976 (0.053)
	(3, 2)	0.1989 1560 (0.080)	0.6366 3440 (0.174)
	(3, 3)	0.3121 2467 (0.116)	0.6875 2533 (0.118)
	(3, 4)	0.3709 3324 (0.131)	0.7275 1676 (0.085)
	(4, 2)	0.1648 908 (0.054)	0.8658 4092 (0.040)
	(4, 3)	0.2717 1676 (0.085)	0.6208 3324 (0.139)
	(4, 4)	0.0384 2481 (0.012)	0.9617 2519 (0.011)

Table V
Estimated Average Values of R and Their Standard Deviations (in Parentheses)

(M, N)	(m, n)	R, W<1	R, W$\geq$1
(2, 2)	(2, 2)	0.2500 2542 (0.122)	0.7500 2458 (0.138)
	(3, 3)	0.3087 2491 (0.123)	0.6884 2509 (0.122)
	(4, 4)	0.0429 2457 (0.0090)	0.9573 2543 (0.0090)

Table VI

Right Tail Values of the F-Distribution with M=N, And the Corresponding Tail Area

(M, N)	.01	.025	.05
(2, 2)	99.00	39.00	19.00
(4, 4)	16.00	9.60	6.39
(6, 6)	8.47	5.82	4.28
(8, 8)	6.03	4.43	3.44
(10, 10)	4.85	3.72	2.98
(12, 12)	4.16	3.28	2.69

The values, in the table VI, show the relationship between the ratio of N/M and ρ. In case M = N, ρ should assume the value in the table in order to have a good value for R. For example, when (M, N) = (2, 2), and for R to be 0.99, ρ should take the value 99.00, and thus the ratio of μ / λ should be that much, that is μ should be 99 times bigger than λ.

Figure 1, BURR TYPE X DISTRIBUTION

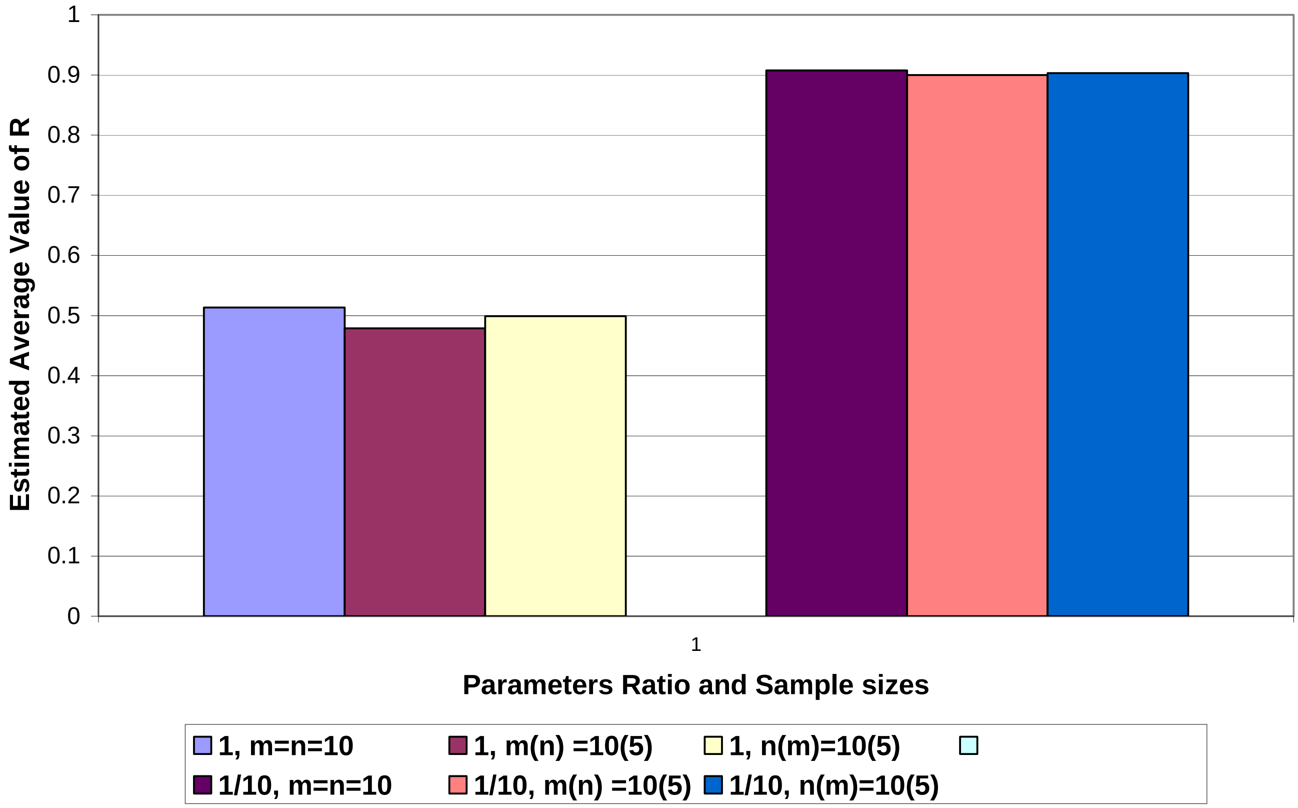


Figure 2. Exponential Distribution, (M, N) = (1, 1)

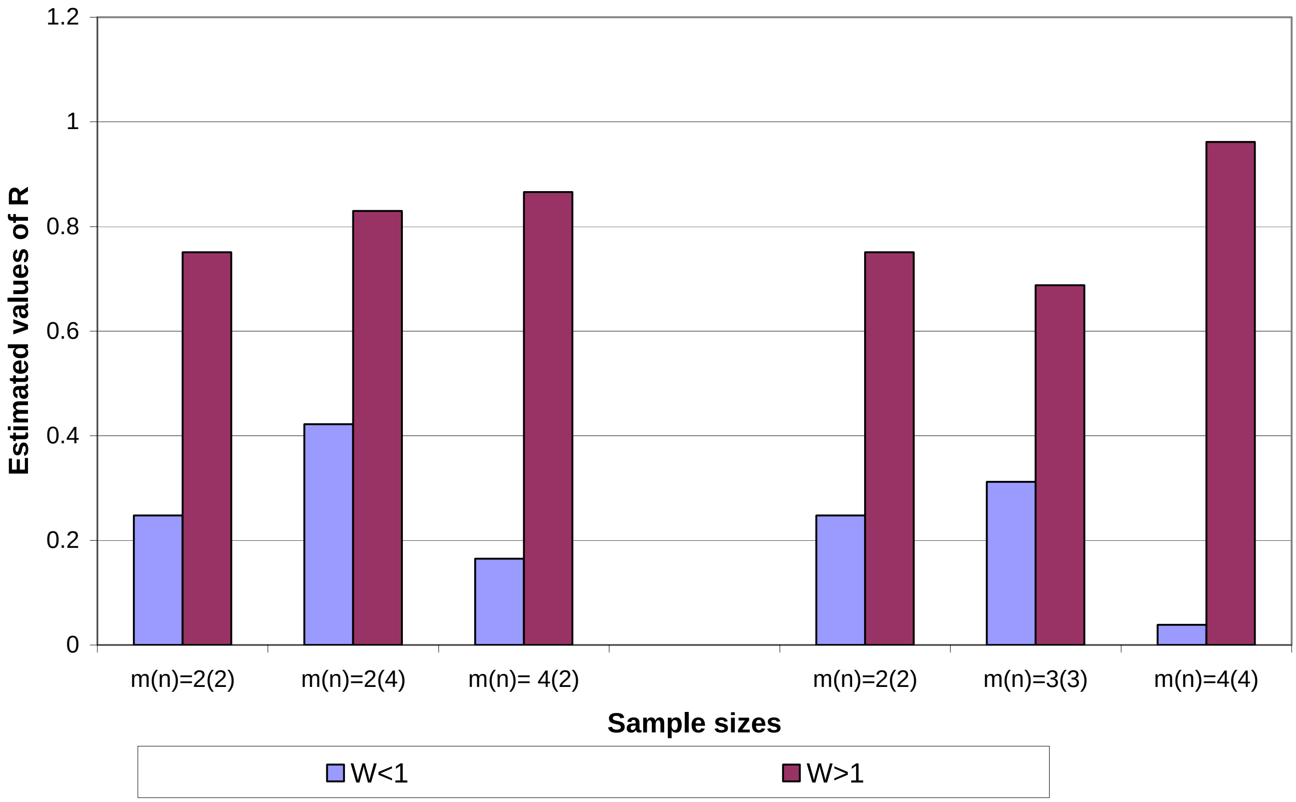
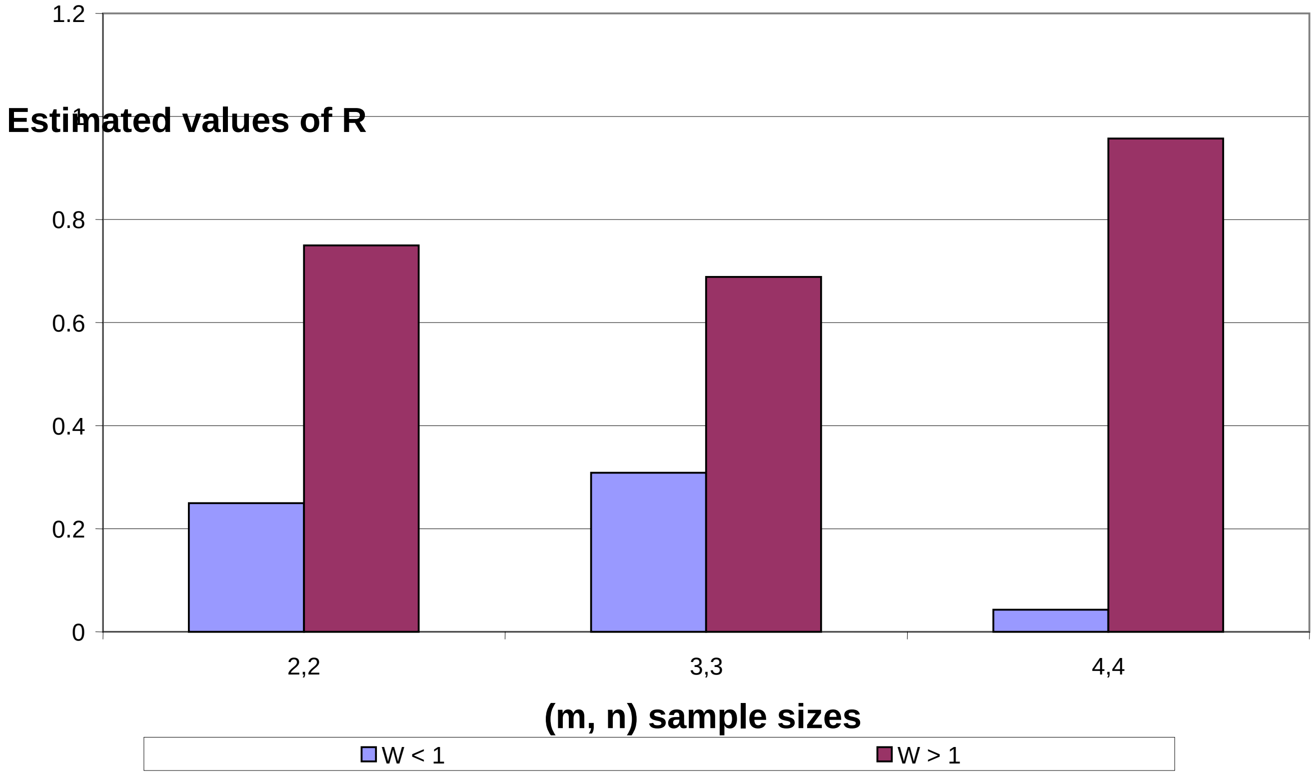


Figure 3 (M, N) = (2, 2) Gamma case



4. CONCLUSIONS

Industry is data rich and yet it is very often that decisions are made based on 2, and in most cases on less than 6, readings. The above distribution was chosen based on the fact that the Burr type *X* distribution is very useful in modeling strength data.

In case of Burr Type X assumption, Tables I, II, and IIII have shown that, aside from not knowing the values of λ&θ, the ratio λ/θ is an important factor in evaluating the reliability of the system. Moreover, it is the ratio λ/θ that mostly determines the value of *R*. The difference in values of the sample sizes for the stress and the strength, for a given ratio, does not affect the estimated value of *R* as long as the ratio of those sample sizes kept fixed. Thus this study shows that it is imperative to have the parameter of the stress much smaller than that of the strength despite that they share the same distribution.

For the Gamma distribution assumption, when M and N increase, and for $w < 1, w^{N+j}$ goes smaller, as well as w^{-M-i} when $w \geq 1$. This implies that increasing M & N and the sample sizes m & n, for $w < 1$, would result in a smaller value for R, while on the other hand, for $w \geq 1$, increasing M & N and m & n would make R larger. Since the means of the Gamma distribution for X & Y are given by Mλ & Nμ respectively, if we keep $\lambda = \mu$, M has to be much larger than N to generate a $w \geq 1$. Moreover, keeping M = N, λ must be much larger than μ in order to generate $w \geq 1$.

Figures 1. 2, and 3 show that the estimation of R is more dependent on the ratio between the populations' parameters than on the samples' sizes

Again as it is completely perceived that the larger the sample the better, is not a good practical assumption for evaluating the reliability of a system. Attention should be paid to the values of the parameters that control that distribution when it is assumed for the stress and strength data. An extension of this work to include more distributions for comparisons is ongoing. The Weibull distribution will be considered as a competitive for the Gamma Distribution.

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