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A Two-node Task-Splitting Feedback Tandem Queueing System with Infinite Buffers

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ABSTRACT

A two-node single-processing tandem queueing system with feedback and task splitting with infinite buffer is considered. In this paper, to solve the stationary queue length distribution we will solve a functional equation and will offer an algorithm to obtain numerical distribution of the stationary queue size.

1. Introduction

Analysis of many stationary queueing systems involving two servers using generating function leads to a functional equation such as $A(z,w)G(z,w) = B_1(z,w)G(0,w) + B_2(z,w)G(z,0) + C(z,w)G(0,0)$ whose unknown is the generating function $G(z, w)$. Solution of such a functional equation reduces to solving a Riemann-Hilbert Problem. Fayolle and Iasnogorodski (1979) are the first to introduce the theory of boundary value problems to the field of queueing theory. They analyzed a two independent parallel $M/M/1$ queueing systems with infinite buffers. See also Fayolle (1979) and Iasnogorodski (1979). Cohen and Boxma (1983) applied the technique to solve some standard queueing models. There are other methods for solving a functional equation. For instance, Jaffe (1992) uses automorphy.

Solution of the functional equation mentioned requires analysis of the function $A(z, w)$. This analysis has been done in the literature. Fayolle et al. (1982) consider steady-state analytic solution of a class of two-dimensional birth-and-death process with “limited state-dependency”. They show that the steady-state distribution of the random walk they consider reduces to the solution of a Hilbert problem on a circle. Although the outstanding problem in this area concerns the ergodicity conditions, they are not able to establish the necessary and sufficient conditions in terms of the parameters of the model they are considering. The authors claim that their method leads to numerical implementation, but they do not show how this is possible while they offer a graphic numerical example. Mikou (1981) studies a two-node Jackson's network in which he considers a functional equation similar to what was mentioned. He has redone the procedure of Fayolle et al. (1982) in more detail. Konheim and Reiser (1976) have studied a two-station cyclic tandem queueing system with finite and infinite buffers. For the infinite case they find the solution as a product-form.

2. Description of the model

We consider a two-node tandem queueing system, Figure1. Each node is a single-processing unit with an infinite-sized buffer before it. Tasks arrive from an infinite source according to a Poisson process with mean ℓ to the first node. No skipping is allowed. After a task completes it processing in either node, it may leave the system with probability q_i, i = 1, 2, or join one of the node with probability p_i^j, i, j = 1, 2. It is also possible that leaving a node, a task goes for split with probability p_{s_i}, i = 1, 2. $q_1 + p_1^1 + p_1^2 + p_1^{s_1} = 1$ and $q_2 + p_2^2 + p_2^1 + p_2^{s_2} = 1$. In that case a task splits into two subtasks. These subtasks become independent, asynchronous tasks which themselves may further split into subtasks thereby giving sub-subtasks, etc. If an outgoing task from a node splits, one subtask must return the note it exited from. The other subtask could leave the system with probability q_{s_i}, i = 1, 2, or join node i with probability $p_{s_i}^j$, i, j = 1, 2. $q_{s_1} + p_1^{s_1} p_{s_1}^1 + p_1^{s_1} p_{s_1}^2 = 1$ and $q_{s_2} + p_2^{s_2} p_{s_2}^2 + p_2^{s_2} p_{s_2}^1 = 1$. The processing distribution in both nodes are exponential with parameters μ_i, i = 1, 2. The processing discipline is first-come-first-served.

We note that a model such as described above without splitting part has been considered in Montazer-Haghighi (1976). Laplace Transform of the generating function is given. No solution is given for it. There has been noted that no method known by then can solve this generating function. Some particular cases have also been discussed. However, it has been noted also that even those particular cases could not be solved. We also note that the same model with a case of both buffers finite and another case with the second buffer finite has been considered in Haghighi and Mishev (2007).

Let the random variables $\xi_i(t)$, i = 1, 2, denote the number of tasks in node i at time t, including the ones being processed. Let also the joint probability of m_1, $0 \le m_1 \le \infty$, tasks at the first node, including the one being processed, and m_2, $0 \le m_2 \le \infty$, tasks at the second node 2, including the ones being processed, at time t be denoted by $P_{m_1,m_2}(t)$, i.e., $P_{m_1,m_2}(t) = P\{\xi_1(t) = m_1, \xi_2(t) = m_2\}$. Further, let $P_{m_1,m_2} = \lim_{t\to\infty} P_{m_1,m_2}(t)$ be the steady-state probability of having m_1 tasks in node 1 and m_2 tasks in node 2, i.e., $m_1 + m_2$ tasks in the system.

It should be noted that as in many cases in the literature, we realize that our model is yet idealized since the Poisson arrival, the stochastic routing rule, the exponential servers and considering only one type of job may all be challenged. We however, feel that such even idealized model will provide foundation for models with approximate solutions.

3. Analysis

Now suppose that each node acts independently of the other. Then assuming that arrival rates to the nodes are ℓ_1 and ℓ_2, and considering the stationary process, the output mean rate of the first node is ℓ_1 and of the second node is ℓ_2. See Burke (1956) or Takács (1962, p. 45). From Figure

1, it can be seen that arrival rates $\lambda_i, i = 1, 2$ to nodes 1 and 2 can be found from the system of linear equations

$$\begin{cases} \lambda_1 = \lambda + \left[p_1^1 + p_1^{s_1}\left(1 + p_{s_1}^1\right)\right]\lambda_1 + \left(p_2^1 + p_2^{s_2} p_{s_2}^1\right)\lambda_2 \\ \lambda_2 = \left(p_1^2 + p_1^{s_1} p_{s_1}^2\right)\lambda_1 + \left[p_2^2 + p_2^{s_2}\left(1 + p_{s_2}^2\right)\right]\lambda_2 \end{cases}$$

as follows:

$$\lambda_1 = \frac{\left[1 - p_2^2 - p_2^{s_2}\left(1 + p_{s_2}^2\right)\right]\lambda}{\left[1 - p_1^1 - p_1^{s_1}\left(1 + p_{s_1}^1\right)\right]\left[1 - p_2^2 - p_2^{s_2}\left(1 + p_{s_2}^2\right)\right] - \left(p_1^2 + p_1^{s_1} p_{s_1}^2\right)\left(p_2^1 + p_2^{s_2} p_{s_2}^1\right)}$$

(1)

and

$$\lambda_2 = \frac{\left(p_1^2 + p_1^{s_1} p_{s_1}^2\right)\lambda}{\left[1 - p_1^1 - p_1^{s_1}\left(1 + p_{s_1}^1\right)\right]\left[1 - p_2^2 - p_2^{s_2}\left(1 + p_{s_2}^2\right)\right] - \left(p_1^2 + p_1^{s_1} p_{s_1}^2\right)\left(p_2^1 + p_2^{s_2} p_{s_2}^1\right)}.$$

(2)

Note that for a simple *M*/*M*/1 queue with feedback from (1) we will have $\lambda_1 = \dfrac{\lambda}{1 - p_1^1}$, as expected. It should also be noted that in this case the solution exists if and only if $\rho_1 = \dfrac{\lambda_1}{\mu_1} < 1$, $\rho_2 = \dfrac{\lambda_2}{\mu_2} < 1$.

The system of balance equations for our model is as follows:

$$\lambda P_{0,0} = q_1\mu_1 P_{1,0} + q_2\mu_2 P_{0,1}$$

$$\{\lambda + [q_1 + p_1^2 + p_1^{S_1}(p_{S_1}^1 + p_{S_1}^2)]\mu_1\}P_{1,0} = \lambda P_{0,0} + q_2\mu_2 P_{1,1} + p_2^1\mu_2 P_{0,1} + q_1\mu_1 P_{2,0}$$

$$\{\lambda + [q_1 + p_1^2 + p_1^{S_1}(p_{S_1}^1 + p_{S_1}^2)]\mu_1\}P_{m_1,0} = (\lambda + p_1^{S_1} p_{S_1}^1\mu_1)P_{m_1-1,0} + q_1\mu_1 P_{m_1+1,0} + q_2\mu_2 P_{m_1,1} + p_2^1\mu_2 P_{m_1-1,1}, m_1 \ge 2$$

$$\{\lambda + [q_2 + p_2^1 + p_2^{S_2}(p_{S_2}^2 + p_{S_2}^1)\mu_2]\}P_{0,1} = p_1^2\mu_1 P_{1,0} + q_1\mu_1 P_{1,1} + q_2\mu_2 P_{0,2}$$

$$\{\lambda + [q_2 + p_2^1 + p_2^{S_2}(p_{S_2}^2 + p_{S_2}^1)]\mu_2\}P_{0,m_2} = q_1\mu_1 P_{1,m_2} + p_1^2\mu_1 P_{1,m_2-1} + p_2^{S_2} p_{S_2}^2\mu_2 P_{0,m_2-1} + q_2\mu_2 P_{0,m_2+1}, \quad m_2 \ge 2$$

$$\{\lambda + [q_1 + p_1^2 + p_1^{S_1}(p_{S_1}^1 + p_{S_1}^2)]\mu_1 + [q_2 + p_2^1 + p_2^{S_2}(p_{S_2}^1 + p_{S_2}^2)]\mu_2\}\, P_{1,1}$$
$$= (\lambda + p_2^{S_2} p_{S_2}^1\mu_2)P_{0,1} + p_1^2\mu_1 P_{2,0} + p_2^1\mu_2 P_{0,2} + q_1\mu_1 P_{2,1} + q_2\mu_2 P_{1,2} + p_1^{S_1} p_{S_1}^2\mu_1 P_{1,0}$$

$$\{\lambda + [q_1 + p_1^2 + p_1^{S_1}(p_{S_1}^1 + p_{S_1}^2)]\mu_1 + [q_2 + p_2^1 + p_2^{S_2}(p_{S_2}^1 + p_{S_2}^2)]\mu_2\}\, P_{m_1,1}$$
$$= p_1^{S_1} p_{S_1}^2\mu_1 P_{m_1,0} + p_1^2\mu_1 P_{m_1+1,0} + (\lambda + p_1^{S_1} p_{S_1}^1\mu_1 + p_2^{S_2} p_{S_2}^1\mu_2)P_{m_1-1,1} + p_2^1\mu_2 P_{m_1-1,2}$$
$$+ q_1\mu_1 P_{m_1+1,1} + q_2\mu_2 P_{m_1,2},\ m_1 \ge 2$$

$$\{\lambda+[(q_1+p_1^2+p_1^{S_1}(p_{S_1}^1+p_{S_1}^2)]\mu_1+[q_2+p_2^1+p_2^{S_2}(p_{S_2}^1+p_{S_2}^2)]\mu_2]\}P_{1,m_2}$$
$$=(\lambda+p_2^{S_2}p_{S_2}^1\mu_2)P_{0,m_2}+p_2^1\mu_2P_{0,m_2+1}+(p_1^{S_1}p_{S_1}^2\mu_1+p_2^{S_2}p_{S_2}^2\mu_2)P_{1,m_2-1}+p_1^2\mu_1P_{2,m_2-1}$$
$$+q_1\mu_1P_{2,m_2}+q_2\mu_2P_{1,m_2+1},\quad m_2\geq 2$$

$$\{\lambda+(q_1+p_1^2+p_1^{S_1}(p_{S_1}^1+p_{S_1}^2)\mu_1+[q_2+p_2^1+p_2^{S_2}(p_{S_2}^1+p_{S_2}^2)]\mu_2]\}P_{m_1,m_2} \tag{3}$$
$$=p_2^1\mu_2P_{m_1-1,m_2+1}+(\lambda+p_1^{S_1}p_{S_1}^1\mu_1+p_2^{S_2}p_{S_2}^1\mu_2)P_{m_1-1,m_2}+(p_1^{S_1}p_{S_1}^2\mu_1+p_2^{S_2}p_{S_2}^2\mu_2)P_{m_1,m_2-1}$$
$$+p_1^2\mu_1P_{m_1+1,m_2-1}+q_1\mu_1P_{m_1+1,m_2}+q_2\mu_2P_{m_1,m_2+1},\ m_1,m_2\geq 2$$

with $q_1+p_1^1+p_1^2+p_1^{s_1}=1, q_{s_1}+p_{s_1}^1+p_{s_1}^2=1, q_2+p_2^2+p_2^1+p_2^{s_2}=1,$ and $q_{s_2}+p_{s_2}^1=1.$ The normalization is provided by

$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}P_{m_1,m_2}=1. \tag{4}$$

There are different methods for solving system of equation such as (3). For instance, Konheim and Reiser (1976, 1978) use generating functions, Latouche and Nuets (1980) use matrix analytic method, and Grassmann and Drekic (2000) use eigenvalues method. However, in many cases there are no closed form solutions. In this paper, we will apply the standard generating function method, obtain a functional equation, and then based on available discussions in the literature we will offer an algorithmic solution.

Let

$$G(z,w)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}P_{m_1,m_2}z^{m_1}w^{m_2},\quad |z|\leq 1,\ |w|\leq 1, \tag{5}$$

with

$$G(z,0)=\sum_{m_1=0}^{\infty}P_{m_1,0}z^{m_1},\quad |z|\leq 1, \tag{6}$$

and

$$G(0,w)=\sum_{m_2=0}^{\infty}P_{0,m_2}w^{m_2},\quad |z|\leq 1\ . \tag{7}$$

Applying (5) – (7) on (3) leads to the functional equation

$$A(z,w)G(z,w)=B_1(z,w)G(0,w)+B_2(z,w)G(z,0), \tag{8}$$

where

$$A(z,w)=[(p_1^{s_1}p_{s_1}^2\mu_1+p_2^{s_2}p_{s_2}^2\mu_2)z+p_1^2\mu_1]w^2+\{(\lambda+p_1^{s_1}p_{s_1}^1\mu_1+p_2^{s_2}p_{s_2}^1\mu_2)z^2$$
$$-[\lambda+(q_1+p_1^{s_1}p_{s_1}^1+p_1^{s_1}p_{s_1}^2+p_1^2)\mu_1+(q_2+p_2^1+p_2^{s_2}p_{s_2}^1+p_2^{s_2}p_{s_2}^2)\mu_2]z+q_1\mu_1\}w$$
$$+\mu_2 z(p_2^1 z+q_2),$$
(9)
$$B_1(z,w)=\mu_1 w\{(p_1^{s_1}p_{s_1}^2 z+p_1^2)w-[q_1+p_1^{s_1}p_{s_1}^1(1-z)+p_1^{s_1}p_{s_1}^2+p_1^2]z+q_1\},$$
(10)

and
$$B_2(z,w)=\mu_2 z\{p_2^{S_2}p_{S_2}^2 w^2-[(q_2+p_2^1+p_2^{S_2}p_{S_2}^2)+p_2^{S_2}p_{S_2}^1(1-z)]w+p_2^1 z+q_2\}.$$
(11)

Remarks:

1. Equation (8) says that $G(z,w)$ is determined by its boundary values $G(z,0)$, $|z|\le 1$, and $G(0,w)$, $|w|\le 1$. Setting z or w equal to zero yields identity. Thus, (8) imposes no restriction on $G(z, 0)$ and $G(0,w)$, though $G(z,w)$ must be analytic for $|z|,|w|<1$ and continuous for $|z|,|w|\le 1$. Therefore, the right hand side of (8) must be zero when $A(z, w)$ is with $|z|,|w|\le 1$. It can be seen that this condition is sufficient to determine $G(z,0)$ and $G(0,w)$ and thus $G(z,w)$ is uniquely determined up to a multiplicative constant. Our focus, therefore, will be on finding $G(z,0)$ and $G(0,w)$.

2. The functional equation (8) is similar to the one Fayolle and Iasnogorodski (1979) obtained from a completely different model, of course with different coefficients. They go through a very detailed analysis of this equation. We, in this paper, will apply some of their results to our case.

3. By Hartog's Theorem the functions $A(z,w)$, $B_1(z,w)$ and $B_2(z,w)$ are analytic since each is a polynomial in each variable z or w, see Shabat (1992).

4. Since $G(z,w)$ is a probability generating function, it is finite. For $G(z,w)$ to exist, from (6) we see that the numerator must vanish whenever $A(z, w)$ does in $|z|<1$ and $|w|<1$. It is, therefore, necessary to examine the algebraic curve defined by $A(z, w) = 0$. $A(z, w)$ is a third degree polynomial with respect of z and w. However, it is a second degree polynomial in each z and w alone. Note that $A(1,1)=B_1(1,1)=B_2(1,1)=0$, $B_1(z, 0)=B_2(0, w)=0$, $A(0, 0)=0$, $G(0, 0)=0/0$. Thus, zeros of $G(z, w)$ are the same as its poles.

5. Ergodicity of the system makes it easy to compute $G(0,1)$ and $G(1,0)$. Replacing z by 1 in (8) the factor $(w-1)$ cancels. Setting $w = 1$ then gives
$$(-q_2\mu_2+p_1^2\mu_1-p_2^1\mu_2+p_1^{s_1}p_{s_1}^2\mu_1+p_2^{s_2}p_{s_2}^2\mu_2)=(p_1^2+p_1^{s_1}p_{s_1}^2)\mu_1 G(0,1)-(q_2+p_2^1)\mu_2 G(1,0).$$
(12)
Similarly, we obtain

$$(\lambda - q_1\mu_1 - p_1^2\mu_1 + p_2^1\mu_2 + p_1^{s_1} p_{s_1}^1 \mu_1 + p_2^{s_2} p_{s_2}^2 \mu_2) = (-q_1 - p_1^2 + p_1^{s_1} p_{s_1}^1)\mu_1 G(0,1) + (p_2^1 + p_2^{s_2} p_{s_2}^2)\mu_2 G(1,0).$$

(13)

This linear system, (12) and (13) can be solved by the Cramer's Rule.

6. $A(w,z)$ defined in (9) is the kernel of the functional equation (8). This kernel is, for each z, a polynomial equation of degree two in w. Thus, for each z there are two possible values of w, say, $\omega_1(z)$ and $\omega_2(z)$, such that $A(z,\omega_1(z)) = A(z,\omega_2(z)) = 0$. Hence, we now will have the following lemmas similar to those of Resing and Örmeci (2003) and Fayolle and Iasnogorodski (1979) that will lead to the solution of (8).

Lemma 1: The algebraic function $\omega \equiv \omega(z)$ defined by $A(z,\omega(z)) = 0$ has four real branch points $0 < z_1^* < z_2^* \le 1 < z_3^* < z_4^*$.

Proof: Branch points are zeros of the discriminant, $D(z)$, of $A(z,w) = 0$ as a function of z, i.e.,

$$D(z) = \left\{ \begin{array}{l} (\lambda + p_1^{s_1} p_{s_1}^1 \mu_1 + p_2^{s_2} p_{s_2}^1 \mu_2) z^2 - [\lambda + (q_1 + p_1^{s_1} p_{s_1}^1 + p_1^{s_1} p_{s_1}^2 + p_1^2)\mu_1 \\ \qquad + (q_2 + p_2^1 + p_2^{s_2} p_{s_2}^1 + p_2^{s_2} p_{s_2}^2)\mu_2] z + q_1\mu_1 \end{array} \right\}^2$$
$$- 4[(p_1^{s_1} p_{s_1}^2 \mu_1 + p_2^{s_2} p_{s_2}^2 \mu_2) z + p_1^2 \mu_1](p_2^1 z + q_2)\mu_2 z$$

It is clear that (a) $D(0) = (q_1\mu_1)^2 > 0$, (b) $D(1) = \left[(p_1^{s_1} p_{s_1}^2 \mu_1 + p_2^{s_2} p_{s_2}^2 \mu_2) z + p_1^2 \mu_1 \right]^2 > 0$ and (c) $\lim_{z\to\infty} D(z) = \infty$. We will now prove (d) that there are there are two numbers, say z_1^0 and z_2^0, $0 < z_1^0 < 1 < z_2^0$, such that $D(z_1^0) < 0$ and $D(z_2^0) < 0$. To prove (d) let

$$(\lambda + p_1^{s_1} p_{s_1}^1 \mu_1 + p_2^{s_2} p_{s_2}^1 \mu_2) z^2 - [\lambda + (q_1 + p_1^{s_1} p_{s_1}^1 + p_1^{s_1} p_{s_1}^2 + p_1^2)\mu_1 + (q_2 + p_2^1 + p_2^{s_2} p_{s_2}^1 + p_2^{s_2} p_{s_2}^2)\mu_2] z + q_1\mu_1 = 0$$

(14)

The discriminant of the quadratic equation (14) is

$$D_0 = \left[(\lambda + p_1^{s_1} p_{s_1}^1 \mu_1 + p_2^{s_2} p_{s_2}^1 \mu_2) - q_1\mu_1 \right]^2$$
$$+ 2\left[(\lambda + p_1^{s_1} p_{s_1}^1 \mu_1 + p_2^{s_2} p_{s_2}^1 \mu_2) + q_1\mu_1 \right] \cdot \left[q_2\mu_2 + p_1^{s_1} p_{s_1}^2 \mu_1 + p_2^{s_2} p_{s_2}^2 \mu_2 + p_1^2\mu_1 + p_2^1\mu_2 \right]$$
$$+ \left[q_2\mu_2 + p_1^{s_1} p_{s_1}^2 \mu_1 + p_2^{s_2} p_{s_2}^2 \mu_2 + p_1^2\mu_1 + p_2^1\mu_2 \right]^2 > 0$$

Therefore, equation (14) has two real solutions, say z_1^0 and z_2^0. Since $z_1^0 + z_2^0 > 1$ and $z_1^0 \cdot z_2^0 > 0$, $z_1^0 > 0$ and $z_2^0 > 0$. Also since $F(0) > 0$, $F(1) < 0$, and $\lim_{z\to\infty} F(z) = \infty$, we will have $0 < z_1^0 < 1 < z_2^0$. Hence, $D(z_1^0) < 0$ and $D(z_2^0) < 0$. This completes proof of (d).

(a), (b), (c), and (d) imply the existence of four points z_1^*, z_2^*, z_3^* and z_4^* such that $0 < z_1^* < z_2^* \le 1 < z_3^* < z_4^*$ and $D(z_i^*) = 0, i = 1,2,3,4$. This completes the proof of Lemma 1.

Lemma 2: For each $z \in [z_1^*, z_2^*]$ the two roots $\omega_1(z)$ and $\omega_2(z)$ are complex conjugates. Hence, the interval $[z_1^*, z_2^*]$ is mapped by $z \to \omega(z)$ onto a closed contour *L*, which is symmetric with respect to the real line.

Proof: The statement of the lemma follows directly from the facts that $D(z_1) = D(z_2) = 0$ and $D(z) < 0$ for $z_1^* < z < z_2^*$.

Note that if $\bar{\omega}(z)$ indicates the complex conjugate of $\omega(z)$, then $\omega_2(z) = \bar{\omega}_1(z)$ and

$$\omega_1(z) \cdot \omega_2(z) = \omega_1(z)\,\overline{\omega_1(z)} = \frac{(q_2\mu_2 + p_2^1\mu_2 z)z}{(p_1^{s_1} p_{s_1}^2 \mu_1)z + p_1^2\mu_1}.$$

It is clear that *A*(*z*, *w*) is, for each *w*, a polynomial function of degree two in *z*. Thus, for every value of w there are two possible values, say $\xi_1(w)$ and $\xi_2(w)$, such that $A\left(\xi_1(w), w\right) = A\left(\xi_2(w), w\right) = 0$. Thus, the following two lemmas can easily be proved.

Lemma 3: The algebraic function $\xi \equiv \xi(w)$ defined by $A(\xi(w), w) = 0$ has four real branch points $0 < w_1^* < w_2^* \le 1 < w_3^* < w_4^*$.

Lemma 4: For each $w \in [w_1^*, w_2^*]$ the two roots $\xi_1(w)$ and $\xi_2(w)$ are complex conjugates. Hence, the interval $[w_1^*, w_2^*]$ is mapped by $w \to \xi(w)$ onto a closed contour *L*, which is symmetric with respect to the real line.

Let $\omega(z)$ be an algebraic curve defined over the complex plane by $A(z, \omega(z)) = 0$ such that

$$B_1(z,\omega(z))\, G(0,\omega(z)) + B_2(z,\omega(z))\, G(z, 0) = 0. \tag{15}$$

Fayolle and Iasnogorodski (1979) prove that when *z* describes the circle $L \equiv \left\{ z : |z| = \sqrt{\mu_1 / \lambda_1} \right\}$, $\omega(z)$ describes a real line segment $[t_1, t_2]$ such that $0 < t_1 \le \omega(z) \le t_2 < 1$. They also prove that if $\lambda_1 / \mu_1 < 1$, then an analytic continuation for $B_1(z,\omega(z))\, G(z, 0)$ can be constructed at least up to the circle *L*.

Solving equation $B_1[\zeta(w), w] = 0$ for *w* in $|z| < 1$. Let w_1 and w_2 be roots, if exist. Let the zeros of $B_2(z,\omega(z))$ between the unit circle and *L* be denoted by $z_1, z_2, \cdots, z_k$. Define *B*(*t*) as

$$B(t) = i\,\frac{[t - \zeta(w_1)][t - \zeta(w_2)]B_2[t,\omega(t)]}{(t - z_1)(t - z_2)\cdots(t - z_k)B_1[t,\omega(t)]}, \text{ for } z_1^* < t < z_2^*.$$

The function $G(0,\omega(z))$ being real on the cut $[z_1^*, z_2^*]$ yields

$$Re\left[B(z)G(0,\omega(z))\right]=0, \text{ for } z\in[z_1^*, z_2^*],$$
(16)

where $B(z)=i\dfrac{[z-\xi(w_1)][z-\xi(w_2)]B_2(z,\omega(z))}{B_1(z,\omega(z))(z-z_1)(z-z_2)\cdots(z-z_k)}$. For $t=\omega_1(\zeta(t))$, $t\in L_w$, and $\Psi(t)\equiv G(0,t)$ equation (16) implies that

$$Re\left[B(t)\Psi(t)\right]=0, \text{ for } t\in L_w.$$
(17)

Equation (18) is the homogenous Riemann-Hilbert boundary value problem, or Problem H_0, whose *index* is

$$K\equiv\frac{1}{2\pi}\Delta_{L_w}\arg[\overline{B(t)}],$$
(18)

where $\Delta_L \arg f(t)$ is the variation of the argument of the function $f(t)$ along L traversed in the positive sense, Shabat (1992). Thus, $\Delta_L \arg f(t)$ is determined by the number of times $1/B(t)$ crosses the positive real axis as t traverses L. The essence of the Problem H_0 on a domain D is to find an analytic function $\Psi(z)$ such that it is continuous up to the boundary L and satisfies the boundary condition $\text{Re}\left[B(t)\Psi(t)\right]=0$, $t\in L$. The value of the number $\Delta_L \arg f(t)$, known as the *argument principle*, can be found as $2\prec(N-P)$, where N and P are the number of zeros and poles, respectively, of $f(t)$, counting multiplicities, interior to L, Churchill and Brown (1984). The Problem H_0 has a unique solution if and only if $K = 0$, see Gakhov (1966) or Smirnov (1964).

From (8) and (9) we see that for t such that $|t|=1$, the equation $B_1(\zeta(t),t)=0$ and $B_2(\zeta(t),t)=0$ each has a unique solution in the unit circle L_t. Thus, $K = N - P = 0$. This result is in line with the fact that Problem H_0 has a unique solution if and only if $K = 0$. See Gakhov [7] or Smirnov [19].

The general solution of Problem H_0 defined by (17) when $K = 0$ is $\Psi(t)=ce^{F(t)}$, $t\in D_t^+$, in which $F(t)$ for $t\in D_t^+$ on the unit circle $L_t\equiv\{|t|=1\}$ is given as

$$F(t)=\frac{1}{2\pi}\int_{L_u}\frac{\sigma(u)}{u-t}du,$$
(19)

where

$$\sigma(t)\equiv\arg\left(-\frac{a(t)-ib(t)}{a(t)+ib(t)}\right),$$
(20)

$$a(t)=Re\,[B(t)], \qquad \text{and} \qquad b(t)=Im\,[B(t)],$$
(21)

with $[a(t)]^2 + [b(t)]^2 \neq 0$. See Wen [21], for example.

From (19) and the Cauchy Integral Formula, since $\sigma(u)$ is an analytic function on L_u, we will have $F(t) = i\sigma(t)$, see Churchill and Brown [3]. Thus, $\Psi(t) = ce^{i\sigma(t)}$. From (3), $G(0, 0) = P_{0,0}$. Hence, for t on the unit circle C_t and inside D_t we will have

$$G(0, t) = P_{0,0}e^{i[\sigma(t)-\sigma(0)]}. \tag{22}$$

Substituting (22) in (13) yields $G(0, \omega_1(z_c))$ as

$$G(\zeta(t),\ 0) = -\frac{B_1(\zeta(t), t)}{B_2(\zeta(t), t)} P_{0,0}e^{i[\sigma(t)-\sigma(0)]}. \tag{23}$$

Thus, from (6), (22) and (23) the generating function $G(z, w)$ is found as

$$G(z, w) = \frac{B_2(\zeta(t), t)B_1(z, w) - B_1(\zeta(t), t)B_2(z, w)}{B_2(\zeta(t), t)A(z, w)} P_{0,0}e^{i[\sigma(t)-\sigma(0)]},\ w \in D_t^+,\ |z| < 1. \tag{24}$$

Hence, probabilities P_{m_1,m_2} will be obtained as the real part of the term with m_1^{th} power of z and m_2^{th} power of w in $G(z, w)$ of (24). $P_{0,0}$ will be found using equation (11). This completes proof of part b of Theorem.

Algorithm

The following algorithm gives a sequence of steps to find P_{m_1,m_2}, numerically:

1. Choose values of λ_i and μ_i, such that $\rho_i = \frac{\lambda_i}{\mu_i} < 1$, $i = 1, 2$.
2. From equation (9) solve $A(z, w) = 0$ for w. This gives two roots as functions of z, say $\omega_1(z)$ and $\omega_2(z)$.
3. For z on the unit circles, choose the unique root $\omega_i(z)$, $i = 1$ or 2, of $A(z, w) = 0$ such that $\text{Re}(z) \neq 1$ and $|\omega_i(z)| < 1$.

   To do this, choose z, say $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, such that $|z| = 1$. Then try the values of roots in step 2 at this z, choose the one with absolute value less than 1 and call that $\omega(z)$.
4. From equation (9) solve $A(z, w) = 0$ for z. This gives two roots as functions of w, say $\zeta_1(w)$ and $\zeta_2(w)$.
5. Choose one of the solutions, say $\zeta(w)$, from step 4 such that for w on the unit circle we have $w = \omega(\zeta(w))$.

To do this, choose *w*, say as $\frac{\sqrt{3}}{2}+\frac{1}{2}i$, such that $|w|=1$. Then try the values of $\zeta_1(w)$ and $\zeta_2(w)$ in step 4 and choose the one with $w=\omega(\zeta(w))$, call that $\zeta(w)$.

6. Solve equation $B_1[\zeta(w), w]=0$ for w in $|z|<1$. Let w_1 and w_2 be roots, if exist, $[t-\zeta(w_1)][t-\zeta(w_2)]$. Also solve equation $B_2[z,\omega(z)]=0$ for z in. Choose the solutions between $|z|>1$ and $|z|=L$. Call these solutions as $z_1, z_2, \cdots, z_k$.
7. Define $B(t)$ as $B(t)=i\frac{[t-\zeta(w_1)][t-\zeta(w_2)]B_2[t,\omega(t)]}{(t-z_1)(t-z_2)\cdots(t-z_k)B_1[t,\omega(t)]}$, for $|t|=1$. Choose *t*, so that $|t|=1$. To do this, start with $t_0=.9x+i\sqrt{1-.81x^2}$. Then take *n*, say 1000, different values of *x* and calculate average value of $B(t)$.
8. From step 7 and equation (21) find $a(t)=Re\,[B(t)]$ and $b(t)=Im\,[B(t)]$.
9. From step 8 and equation (20) find $\sigma(t)\equiv\arg\left(-\frac{a(t)-ib(t)}{a(t)+ib(t)}\right)$.
10. From step 9 and equation (24) find $G(z, w)$.
11. Expand $G(z, w)$ from step 10 and choose P_{m_1,m_2} as the real part of the term with m_1th power of *z* and m_2th power of *w* in $G(z, w)$.
12. Find $P_{0,0}$ from step 11 and equation (11).

Note that in step 11, when $z = 0$ or $w = 0$, P_{0,m_2} and $P_{m_1,0}$ can be directly found from (24) by Taylor series of a function of variable.

Conclusion

In this paper, using the standard generating function method, we were able to solve a functional equation leading to stationary distribution of the queue size. Although the algorithm clearly leads to the distribution, passing to limit of a bivariate function as $(z,w)\to(0,0)$ does not seem to be an easy task. We mentioned Fayolle et al. [6] that consider steady-state analytic solution of a class of two-dimensional birth-and-death process with "limited state-dependency". As mentioned earlier, the authors claim that their method leads to numerical implementation, but they do not show how this is possible while they offer a graphic numerical example. Had they gone through a detailed numerical example, they could have experience similar difficulties we experienced in this paper.

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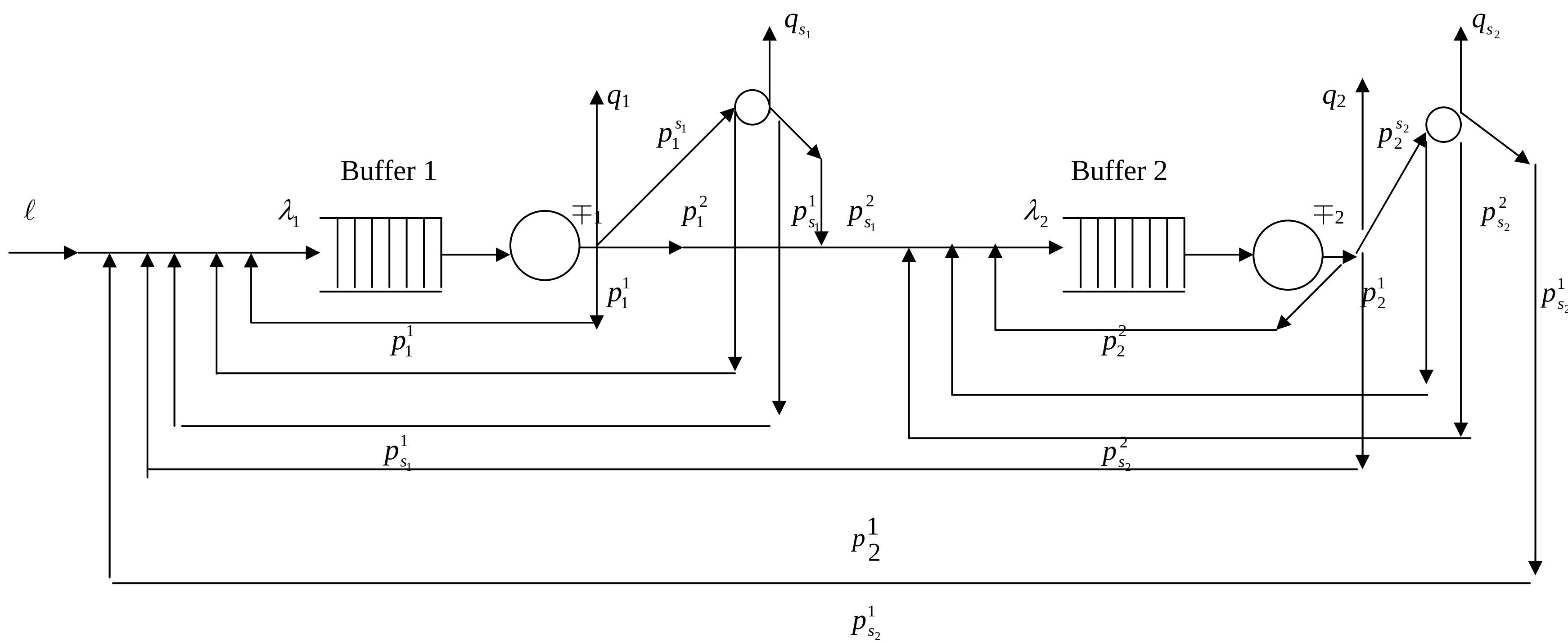

Buffer 1
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Figure 1