DEPARTMENT OF MATHEMATICS
TECHNICAL REPORT
ISSN: 1933-1746

The Representations of the Heisenberg Group over a Finite Field

M. Misaghian

MDTRS No. 11

March 30, 2009

COLLEGE OF ARTS & SCIENCES
PRAIRIE VIEW A&M UNIVERSITY
PRAIRIE VIEW, TEXAS 77446-0519

The Representations of The Heisenberg Group over a Finite Field

Manouchehr Misaghian, Ph.D
Department of Mathematics
Prairie View A & M University
P.O. Box 519, Mail stop 2225
Prairie View, TX 77446
E-Mail: mamisaghian@pvamu.edu

Abstract. We find all irreducible representations of the Heisenberg group over a finite field.
Keywords.Character, irreducible representation, finite field, Heisenberg group
MSC:20G05, 11T99, 20C 35, 22E 70.

1. Introduction.

Let F be a local p field where p is a prime number. Let W be a finite-dimensional vector space over F; and let $\langle,\rangle$ be a non-degenerate symplectic bilinear form on W. Let $G = Sp(W)$ be the group of isometries of W with respect to $\langle,\rangle$. The Weil representation of G which arises in considering unitary representations of the Heisenberg group attached to W, has an importatnt role in Θ-correspondences and automorphic forms. The nontrivial unitary representation theory of $\mathbf{H}$, the Heisenberg group, is given by the following Theorem.

Theorem 1. $(Stone - von\ Neumann)$. *Let χ be a nontrivial character of F^+. Then up to an isomorphism there is only one equivalence class of irreducible unitary representations of* $\mathbf{H}$ *with central character* χ.

In [[5], pages 27-33], there is a general construction for representation of the Heisenberg group when the Characteristic of the residual field of the ground field F, is not 2. In this paper, we will use a specific complete polarization of the symplectic space W, to construct all smooth irreducible representations of the Heisenberg group over a finite field, F_{p^m} including p = 2. Our approach can be applied to infinite cases as well, i.e. when the ground field is a local non-Archimedean p-field; However, to exhaust all representations, we use the finiteness of the ground field (in a finite case, the number of all irreducible representations are known.)

Although a couple of the Lemmas and corollaries are found in the literatures, in this paper, we present them with detailed and simple proofs for convenience and completeness.

2. Heisenberg Group

Let F be a finite field with $q = p^m$ elements; where p is a prime number and m is a positive integer. Let W be a finite-dimensional vector space over F; and let $\langle , \rangle$ be a non-degenerate symplectic bilinear form on W; i.e. $\langle , \rangle$ is a map from $W \times W \to F$ having the following properties:

(1)- $\langle , \rangle$ is linear in each variable.

(2)- $\langle , \rangle$ is non-degenerate; i.e. for any $w \in W, w \neq 0$ there is $w' \in W$ such that $\langle w, w' \rangle \neq 0$.

(3)- $\langle , \rangle$ is symplectic ; i.e. for all $w, w' \in W$ we have $\langle w, w' \rangle = -\langle w', w \rangle$.

The Heisenberg group, $H = H(W)$, attached to W is the group with underying set $W \times F$ and the following multiplication:

$$(w, a)(w', a') = (w + w', a + a' + \langle w, w' \rangle)$$

For all $w, w' \in W$ and $a, a' \in F$. See also [5] and [6].

Let $\dim W = 2n$, for some positive integer n. Thus $|H| = q^{2n+1}$. When $p = 2$ the group is abelian and its representations are all one dimentional (Characters). In this case; we find all representations in the next section.

From now on we assume p is an odd prime number.

Lemma 1. *Let $w \in W, w \neq 0$. Define $\varphi_w : W \to F$ by $\varphi_w(w') = \langle w, w' \rangle$. Then φ_w is onto.*

Proof. Let $b \in F$. If $b = 0$, then take $w' = 0$. So let $b \neq 0$. Since $\langle , \rangle$ is nondegenerate there is $w' \in W$ such that $\langle w, w' \rangle = 1$. Now we have $\varphi_w(bw') = \langle w, bw' \rangle = b\langle w, w' \rangle = b$. □

Lemma 2. *There are $q^{2n} + q - 1$ conjugacy classes in H.*

Proof. Let $(w, 0) \in H$, with $w \neq 0$ Then the conjugacy class that contains this element is

$$(w', a')(w, 0)(-w', -a') = (w, 2\langle w', w \rangle)$$

From here and Lemma 1, we deduce that conjugacy classe of $(w, 0)$ is $\{(w, a) \in H \mid w \neq 0\}$. Thus there are $q^{2n} - 1$ classes of this form and each class contains q elements. Now let $w = 0$ and look for conjugacy classes for $(0, a), a \in F$. Then some computations as above shows that there are q different conjugacy classes of this type with one element in each class. □

Corollary 1. *There are two types of conjugacy classes in* H *with representative elements* $(w,0), w \neq 0$, *and* $(0,a), w \in W, a \in F$.

Corollary 2. *There are* $q^{2n}+q-1$ *irreducible representations of* H. *See* [1], [2], [7]

Lemma 3. *The center of* H, $Z(H)$, *is isomorphic to* F^{+}.

Proof. Let $(w,a) \in Z(H)$, then we must have

$$(w,a)(w',a') = (w',a')(w,a)$$

for all $(w',a') \in H$. Thus

$$(w'+w, a'+a+\langle w',w\rangle) = (w+w', a+a'+\langle w,w'\rangle)$$

From here we must have $\langle w,w'\rangle = \langle w',w\rangle$ or $\langle w,w'\rangle = 0$. Since $\langle,\rangle$ is nondegenerate we get $w=0$. Thus

$$Z(H) = \{(0,a) \in H \mid a \in F\}.$$

Now define $f : Z(H) \to F^{+}$ by $f(0,a) = a$. It is easy to show that f is an isomorphism. See also [5] and [6]. □

Lemma 4. *The group of commutatores of* H, $[H,H]$, *is equal to* $Z(H)$, *the center of* H.

Proof. Let $(w,a), (w',a') \in H$. Then we have

$$(w,a)(w',a')(-w,-a)(-w',-a') = (0,\langle w,w'\rangle).$$

Now apply Lemma 1 to get $[H,H] = Z(H)$. See also [6] □

Corollary 3. *The quotient group* $H/Z(H)$ *is abelian group with* $|H/Z(H)| = \frac{q^{2n+1}}{q} = q^{2n}$ *elements.*

Lemma 5. *Any character (one-dimensional representation)* ρ *of* H *induces a character of* $H/Z(H)$, *and conversely any character of* $H/Z(H)$ *induces a character of* H.

Proof. Let $(w,a), (w',a') \in H$. Then we have:

$$\begin{aligned}
& \rho((w,a)(w',a')(-w,-a)(-w',-a')) \\
= \; & \rho(w,a)\rho(w',a')\rho(-w,-a)\rho(-w',-a') \\
= \; & \rho(w,a)\rho(w',a')(\rho(w,a))^{-1}(\rho(w',a'))^{-1} \\
= \; & 1 \\
= \; & \rho(0,\langle w,w'\rangle)
\end{aligned}$$

Thus ρ is trivial on $[H,H] = Z(H)$. Let $\widetilde{(w,a)}$ be an element of $H/Z(H)$ whose representative is (w,a). Now one can easily check that

$\widetilde{\rho}: H/Z(H) \to \mathbb{C}^{\times}, \widetilde{\rho}\left(\widetilde{(w,a)}\right) = \rho(w,a)$ is a well-defined character of $H/Z(H)$. Conversely any character $\widetilde{\rho}$, of $H/Z(H)$ induces a character of H by $\rho(w,a) = \widetilde{\rho}\left(\widetilde{(w,a)}\right)$. □

Lemma 6. *$H/Z(H)$ is isomorphic to additive group W.*

Proof. Define $\varphi : H \to W$ by $\varphi(w,a) = w$. Then one can check that φ is an onto homomorphis and its kernel is $Z(H)$. □

Corollary 4. *Any character of $H/Z(H)$ will be determined by its value on W. In fact a set of the representatives of $Z(H)$ in $H/Z(H)$ is $W \times \{0\} = \{(w,0)\,|w \in W\}$.*

Proof. Let $\widetilde{(w,a)} \in H/Z(H)$. Then $\widetilde{(w,a)} = \widetilde{(w,0)}$, because $(w,a) - (w,0) = (0,a) \in Z(H)$. □

All characters (One-dimensional representations) of H are given by the following theorem.

Theorem 2. *Let χ be a non-trivial character of F^{+}(See [4] for the existence of a non trivial character of $\mathbb{F}^{+}$.). For any $(w,0) \in W \times \{0\}$ Define $\widetilde{\psi_{(w,\chi)}} : H/Z(H) \to \mathbb{C}^{\times}$ by $\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w',0)}\right) = \chi(\langle w, w' \rangle)$ for $\widetilde{(w',0)} \in H/Z(H)$. Then $\widetilde{\psi_{(w,\chi)}}$ is a character of $H/Z(H)$.*

Proof. First we will show that $\widetilde{\psi_{(w,\chi)}}$ is well defined. Let $\widetilde{(w_1,0)} = \widetilde{(w_2,0)} \in H/Z(H)$ then we must have $(w_1,0) - (w_2,0) \in Z(H)$. From here we have $(w_1 - w_2, 0 + 0 + \langle w_1, w_2 \rangle) \in Z(H)$ Thus $w_1 - w_2 = 0$. Hence $w_1 = w_2 = w'$; i.e.

$$\begin{aligned} \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\right) &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_2,0)}\right) \\ &= \chi(\langle w, w' \rangle). \end{aligned}$$

On the other hand we have:

$$\begin{aligned} \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\widetilde{(w_2,0)}\right) &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1+w_2, 0+0+\langle w_1,w_2\rangle)}\right) \\ &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1+w_2,0)}\right) \\ &= \chi(\langle w, w_1+w_2 \rangle) \\ &= \chi(\langle w, w_1 \rangle + \langle w, w_2 \rangle) \\ &= \chi(\langle w, w_1 \rangle)\,\chi(\langle w, w_2 \rangle) \\ &= \widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_1,0)}\right)\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w_2,0)}\right). \end{aligned}$$

□

Lemma 7. *For any $a \in F^{\times}$ we have $\widetilde{\psi_{(aw,\chi)}} = \widetilde{\psi_{(w,\chi_a)}}$ where $\chi_a(x) = \chi(ax)$ for all $x \in F$ is, a character of F^{+}.*

Proof. For any $\widetilde{(w',0)} \in H/Z(H)$ we have:

$$\begin{aligned}\widetilde{\psi_{(aw,\chi)}}\left(\widetilde{(w',0)}\right) &= \chi(\langle aw, w'\rangle) \\ &= \chi(a\langle w, w'\rangle) \\ &= \chi_a(\langle w, w'\rangle) \\ &= \widetilde{\psi_{(w,\chi_a)}}\left(\widetilde{(w',0)}\right).\end{aligned}$$

□

Lemma 8. *Let χ be a character of F^{+}. If $\widetilde{\psi_{(w,\chi)}} = \widetilde{\psi_{(w',\chi)}}$, then $w = w'$.*

Proof. Let $\widetilde{\psi_{(w,\chi)}} = \widetilde{\psi_{(w',\chi)}}$. Then for any $\widetilde{(w'',0)} \in H/Z(H)$, we have:

$$\begin{aligned}\widetilde{\psi_{(w,\chi)}}\left(\widetilde{(w'',0)}\right) &= \chi(\langle w, w''\rangle) \\ &= \widetilde{\psi_{(w',\chi)}}\left(\widetilde{(w'',0)}\right) \\ &= \chi(\langle w', w''\rangle).\end{aligned}$$

From here we get

$$\chi(\langle w - w', w''\rangle) = 1$$

Since $\langle , \rangle$ is non-degenerate we must have $w - w' = 0$,thus $w = w'$. □

Corollary 5. *There are q^{2n} characters of H.*

Proof. Since $|W \times \{0\}| = |\{(w,0) \,|w \in W\}| = q^{2n}$, the Lemmas 7 and 8 imply that there are q^{2n} characters for $H/Z(H)$. Now apply Lemma 5. □

There are $q - 1$ more irreducible representations of H. These representations have dimensions bigger than one. We determine these representations as follows.

Lemma 9. *Let $B = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_n, \beta_n\}$ be a basis of W having the following properties: (this basis exists because W is a non-degenerate symplectic space.)*

$$\begin{aligned}\langle \alpha_i, \alpha_j\rangle &= \langle \beta_i, \beta_j\rangle = 0, \text{ for all } i,j \\ \langle \alpha_i, \beta_j\rangle &= 0, \text{ for all } i,j, i \neq j \\ \langle \alpha_i, \beta_i\rangle &= 1, \text{ for all } i\end{aligned}$$

Let V be the subspace of W generated by $B_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then:

(1) *For all* $v, v' \in V$, $\langle v, v' \rangle = 0$.
(2) *Let* $w \in W$ *but* $w \notin V$. *Then there exists* $\alpha_k \in B_1$ *such that* $\langle w, \alpha_k \rangle \neq 0$.

Proof. 1. This is a consequence of our choice of the basis B_1.

2. Let $w \in W$. Thus $w = \sum_{i=1}^{n} \lambda_i \alpha_i + \sum_{j=1}^{n} \mu_j \beta_j$, where $\lambda_i, \mu_j \in F$ and $\mu_k \neq 0$ for at least one $k, 1 \leq k \leq n$, because $w \notin V$. From here for this k we have:

$$
\begin{aligned}
\langle w, \alpha_k \rangle &= \left\langle \sum_{i=1}^{n} \lambda_i \alpha_i + \sum_{j=1}^{n} \mu_j \beta_j, \alpha_k \right\rangle \\
&= \sum_{i=1}^{n} \lambda_i \langle \alpha_i, \alpha_k \rangle + \sum_{j=1}^{n} \mu_j \langle \alpha_i, \alpha_k \rangle \\
&= 0 + \mu_k \\
&= \mu_k
\end{aligned}
$$

□

Corollary 6. *Let* W' *be the subspace of* W *generated by* $B_2 = \{\beta_1, \beta_2, \cdots, \beta_n\}$. *For any* $a \in F$ *and* $w \in W'$, $w \neq 0$, *there is* $v \in V$ *such that* $\langle w, v \rangle = a$.

Proof. By Lemma 1 there is $w' \in W$ such that $\langle w, w' \rangle = a$. Now write $w' = v_1 + v_2$ for some $v_1 \in V$ and $v_2 \in W'$(note that we have $W = V \bigoplus W'$). From here and properties of the basis B_2 we have:

$$
\begin{aligned}
a &= \langle w, w' \rangle \\
&= \langle w, v_1 + v_2 \rangle \\
&= \langle w, v_1 \rangle + \langle w, v_2 \rangle \\
&= \langle w, v_1 \rangle + 0 \\
&= \langle w, v_1 \rangle
\end{aligned}
$$

Now set $v = v_1$. □

Lemma 10. *Let* V *be the subspace of* W *introduced in the Lemma 9. Set* $K = \{(v, a) \in H \mid v \in V, a \in F\}$. *Then* K *is a normal abelian subgroup of* H.

Proof. Let (v_1, a_1) and $(v_2, a_2) \in K$. Then by using first part of Lemma 9 we have:

$$
\begin{aligned}
(v_1, a_1)(-v_2, -a_2) &= (v_1 - v_2, a_1 - a_2 + \langle v_1, -v_2 \rangle) \\
&= (v_1 - v_2, a_1 - a_2) \in K
\end{aligned}
$$

Since $K \neq \varnothing$ thus it is a subgroup of H. Also note that we have

$$\begin{aligned}(v_1, a_1)(v_2, a_2) &= (v_1 + v_2, a_1 + a_2) \\ &= (v_2 + v_1, a_2 + a_1) \\ &= (v_2, a_2)(v_1, a_1)\end{aligned}$$

So K is abelian. Now let $(w, a) \in H$. Then for any $(v, b) \in K$ we have

$$\begin{aligned}(w, a)(v, b)(-w, -a) &= (w + v - w, a + b - a + \langle w, v\rangle + \langle w + v, -w\rangle) \\ &= (v, b + \langle w, v\rangle) \in K.\end{aligned}$$

Thus K is normal. □

Lemma 11. *Let χ be a nontrivial character of F^+. Define $\psi : K \to \mathbb{C}^\times$ by $\psi(v, a) = \chi(a)$. Then ψ is a character of K.*

Proof. Let (v_1, a_1) and $(v_2, a_2) \in K$. Then we have:

$$\begin{aligned}\psi((v_1, a_1)(v_2, a_2)) &= \psi\left(v_1 + v_2, a_1 + a_2 + \frac{1}{2}\langle v_1, v_2\rangle\right) \\ &= \psi(v_1 + v_2, a_1 + a_2 + 0) \\ &= \chi(a_1 + a_2) \\ &= \chi(a_1)\chi(a_2) \\ &= \psi(v_1, a_1)\psi(v_2, a_2).\end{aligned}$$

□

Corollary 7. *There are at least $q - 1$ nontrivial character of K.*

Proof. Because there are $q - 1$ nontrivial character of F^+. □

Lemma 12. *Let χ be a nontrivial character of F^+ and ψ, K as in the Lemma 11. Let*

$$\mathcal{C}(H, K) = \{f : H \to \mathbb{C} \mid f(kh) = \psi(k) f(h), \text{ for all } k \in K, h \in H\}.$$

Then $\mathcal{C}(H, K)$ is a vector space over $\mathbb{C}$ of dimension q^n.

Proof. It is easy to show $\mathcal{C}(H, K)$ is a vector space. We will show that $\dim \mathcal{C}(H, K) = q^n$. Let $S = \{s_1, s_2, \cdots, s_{q^n}\}$ be a set of representatives of cosets of K in H. Thus for any $h \in H$ there exist a unique $k \in K$ and $s \in S$ such that $h = ks$. Now for each $i, 1 \leq i \leq q^n$, define $f_i : H \to \mathbb{C}$ as follows

$$f_i(ks_j) = \chi(k)\delta_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

is Kronoker delta. Since for any $f \in \mathcal{C}(H,K)$, we then have

$$f = \sum_{i=1}^{q^n} f(s_i) f_i$$

and if we set $\sum_{i=1}^{q^n} \lambda_i f_i = 0$ we get $\lambda_i = 0$, for all $i, 1 \leq i \leq q^n$, the set of all these functions, $\{f_i\}_i, 1 \leq i \leq q^n$ is a basis for $\mathcal{C}(H,K)$. Thus $\dim \mathcal{C}(H,K) = q^n$. □

Theorem 3. *Notations are as in the Lemmas 12 and 8. Define* $\rho : H \to GL(\mathcal{C}(H,K))$ *by*

$$(\rho(h) f)(h') = f(h'h), \qquad \text{for all } f \in \mathcal{C}(H,K), \text{ and } h, h' \in H.$$

Then $(\rho, \mathcal{C}(H,K))$ *is an irreducible representation of* H *of degree* q^n.

Proof. Let $h_1, h_2 \in H$. We then for all $f \in \mathcal{C}(H,K)$ and $h' \in H$ have:

$$\begin{aligned} (\rho(h_1 h_2) f)(h') &= f((h'h_1) h_2) \\ &= (\rho(h_2) f)(h'h_1) \\ &= \rho(h_1)(\rho(h_2) f)(h') \end{aligned}$$

i.e. $\rho(h_1 h_2) = \rho(h_1)\rho(h_2)$. Thus ρ is a representation of H. To Show $(\rho, \mathcal{C}(H,K))$ is irreducible; it is enough to show that $\psi^h \neq \psi$ for all $h \in H$ and $h \notin K$, where ψ^h is defined by

$$\psi^h(x) = \psi\left(h^{-1} x h\right), \quad \text{for all } x \in K.$$

See [7]. Since χ is nontrivial there is $a \in K$ such that $\psi(v, a) = \chi(a) \neq 1$, for all $v \in V$. (V is the same as in the Lemma 9.) Now Let W' be the same as in the Corollary 6. Let $h = (v, b) \in H \backslash K$; thus there are $v_1 \in V$ and $v_2 \in W', v_2 \neq 0$, such that $v = v_1 + v_2$.Then by the Lemma 1 there is some $v' \in V$ such that $\langle v_2, v' \rangle = a$. Now let $x = (v', 0) \in K$. We then have:

$$\begin{aligned} \psi^h(x) &= \psi\left(h^{-1} x h\right) \\ &= \psi((-v, -b)(v', 0)(v, b)) \\ &= \psi(v', \langle v, v' \rangle) \\ &= \chi(\langle v, v' \rangle). \end{aligned}$$

On the other hand we have

$$\begin{aligned}\langle v, v'\rangle &= \langle v_1 + v_2, v'\rangle \\ &= \langle v_1, v'\rangle + \langle v_2, v'\rangle \\ &= 0 + \langle v_2, v'\rangle \\ &= \langle v_2, v'\rangle \\ &= a.\end{aligned}$$

From here we get $\psi^h(x) = \chi(\langle v, v'\rangle) = \chi(a) \neq 1$, but $\psi(x) = \psi(v', 0) = \chi(0) = 1$, i.e. $\psi^h(x) \neq \psi(h^{-1}xh)$. See also [5] for another proof method of irreducibility a representation. □

Corollary 8. *Notations are as in the Theorem 3. The central character of* $(\rho, \mathcal{C}(H, K))$ *is* χ.

Proof. Let $h = (0, a)$ be an element in $Z(H\ )$,the center of H. Then for all $f \in \mathcal{C}(H, K)$ and $(w, b) \in H$ we have:

$$\begin{aligned}(\rho(0, a) f)(w, b) &= f((w, b)(0, a)) \\ &= f((0, a)(w, b)) \\ &= \chi(a) f(w, b).\end{aligned}$$

i.e. $\rho(0, a) f = \chi(a) f$, for all $f \in \mathcal{C}(H, K)$. Thus $\rho_{|Z(H\ )} = \chi$. □

Corollary 9. *Different characters of* F^+ *induce different* ρ*'s.*

Corollary 10. *Any irreducible representation of* H *is either a character as in Corollary 5 or a* q^n*- dimensional representation as in Theorem 3.*

Proof. The dimensions of irreducible representations of H, say $n_1, n_2, \cdots, n_k$, where k is the number of the irreducible representations of H must satisfy the following equation:

$$\begin{aligned}|H| &= q^{2n+1} \\ &= \sum_{i=1}^{k} n_i^2.\end{aligned}$$

From Corollary 5 we know there are q^{2n} one dimensional representations of H and from Theorem 3 we get $(q-1)$ irreducible representations of H of dimension q^n . Now note that:

$$\begin{aligned}\sum_{i=1}^{q-1} (q^n)^2 + \sum_{i=1}^{q^{2n}} 1^2 &= (q-1)\, q^{2n} + q^{2n} \\ &= q^{2n+1} \\ &= |H|.\end{aligned}$$

□

3. Representations of H when $p = 2$

Let $p = 2$ and let V, W' be as in Lemma 9 and corollary 6. Thus $W = V \oplus W'$.

Lemma 13. *For any $w \in W$ we have $\langle w, w\rangle = 0$.*

Proof. Write $w = w_1 + w_2$, for $w_1 \in V$ and $w_2 \in W'$. We then have:

$$\begin{aligned}\langle w, w\rangle &= \langle w_1 + w_2, w_1 + w_2\rangle \\ &= \langle w_1, w_1\rangle + \langle w_1, w_2\rangle + \langle w_2, w_1\rangle + \langle w_2, w_2\rangle \\ &= 0 + 2\langle w_1, w_2\rangle + 0 \\ &= 0.\end{aligned}$$

□

Theorem 4. *Let χ be a nontrivial character of F^+. For each $w = w_1 + w_2 \in W$; define $\varphi_{(w,\chi)} : H \to \mathbb{C}^\times$ by*

$$\varphi_{(w,\chi)}(w', a) = \chi(a)\,\chi(\langle w_1, w_2'\rangle)\,\chi(\langle w_1', w_2'\rangle)\,\chi(\langle w_2, w_1'\rangle).$$

For any $(w', a) \in H$, where $w' = w_1' + w_2' \in W$, $w_1' \in V$ and $w_2' \in W'$. Then $\varphi_{(w,\chi)}$ is a character of H whose restriction to F is χ.

Proof. Let (w', a) and $(w'', b) \in H$ and write $w' = w_1' + w_2'$ and $w'' = w_1'' + w_2''$ for $w_1', w_1'' \in V$ and $w_2', w_2'' \in W'$. Then we have:

$$\begin{aligned}(w', a)(w'', b) &= (w' + w'', a + b + \langle w', w''\rangle) \\ &= (w_1' + w_2' + w_1'' + w_2'', a + b + \langle w_1' + w_2', w_1'' + w_2''\rangle) \\ &= ((w_1' + w_1'') + (w_2' + w_2''), a + b + \langle w_1', w_1''\rangle + \langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle + \langle w_2', w_2''\rangle) \\ &= ((w_1' + w_1'') + (w_2' + w_2''), a + b + \langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle).\end{aligned}$$

From here we have:

$$\begin{aligned}&\varphi_{(w,\chi)}(w', a)(w'', b) \\ =\ &\varphi_{(w,\chi)}((w_1' + w_1'') + (w_2' + w_2''), a + b + \langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle) \\ =\ &\chi(a + b + \langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle)\,\chi(\langle w_1, w_2' + w_2''\rangle)\,\chi(\langle w_1' + w_1'', w_2' + w_2''\rangle)\,\chi(\langle w_2, w_1' + w_1''\rangle) \\ =\ &\chi(a + b)\,\chi(\langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle)\,\chi(\langle w_1, w_2' + w_2''\rangle)\,\chi(\langle w_1' + w_1'', w_2' + w_2''\rangle)\,\chi(\langle w_2, w_1' + w_1''\rangle) \\ =\ &\chi(a + b)\,\chi(\langle w_1, w_2' + w_2''\rangle)\,\chi(\langle w_2, w_1' + w_1''\rangle)\,\chi(\langle w_1', w_2''\rangle + \langle w_2', w_1''\rangle)\,\chi(\langle w_1' + w_1'', w_2' + w_2''\rangle) \\ =\ &\chi(a + b)\,\chi(\langle w_1, w_2' + w_2''\rangle)\,\chi(\langle w_2, w_1' + w_1''\rangle)\,\chi(\langle w_1', w_2'\rangle + \langle w_1'', w_2''\rangle) \\ =\ &\chi(a)\,\chi(\langle w_1, w_2'\rangle)\,\chi(\langle w_1', w_2'\rangle)\,\chi(\langle w_2, w_1'\rangle)\,\chi(b)\,\chi(\langle w_1, w_2''\rangle)\,\chi(\langle w_1'', w_2''\rangle)\,\chi(\langle w_2, w_1''\rangle) \\ =\ &\varphi_{(w,\chi)}(w', a)\,\varphi_{(w,\chi)}(w'', b).\end{aligned}$$

Thus $\varphi_{(w,\chi)}$ is a character of H. Inparticular we have: $\varphi_{(w,\chi)}(0,a) = \chi(a)$. □

Corollary 11. *For any $w \in W$ and any two distinct characters of F^{+}, χ_1 and χ_2 we have $\varphi_{(w,\chi_1)} \neq \varphi_{(w,\chi_2)}$.*

Lemma 14. *Let χ be a non-trivial character of F^{+}.Then for any two elements w and w' of W we have $\varphi_{(w,\chi)} \neq \varphi_{(w',\chi)}$.*

Proof. Let $w = w_1 + w_2 \in W$ and $w' = w'_1 + w'_2 \in W$ where $w_1, w'_1 \in V$ and $w_2, w'_2 \in W'$. Suppose $\varphi_{(w,\chi_1)} = \varphi_{(w,\chi_2)}$. For any $v \in V$ and $a \in F$ we then must have

$$\varphi_{(w,\chi_1)}(v,a) = \varphi_{(w,\chi_2)}(v,a)$$

But for the left hand side of this equation we have:

$$\begin{aligned} \varphi_{(w,\chi_1)}(v,a) &= \chi(a)\,\chi(\langle w_1, 0\rangle)\,\chi(\langle v, 0\rangle)\,\chi(\langle w_2, v\rangle) \\ &= \chi(a)\,\chi(\langle w_2, v\rangle). \end{aligned}$$

The same computations gives $\varphi_{(w,\chi_2)}(v,a) = \chi(a)\,\chi(\langle w'_2, v\rangle)$. From here we must have $\chi(\langle w_2, v\rangle) = \chi(\langle w'_2, v\rangle)$ for all $v \in V$. This and the Lemma 9 force to get $w_2 = w'_2$. The same argument by choosing $v \in W'$ gives $w_1 = w'_1$. □

Corollary 12. *The Theorem 4 gives $(q-1)\,q^{2n}$ characters of H.*

Proof. This is a consequence of the corollary 11 and the lemma 14. □

The rest of the characters of H are given in the following theorem.

Theorem 5. *Let χ be a nontrivial character of F^{+}; and let $K = \{0\}\times F$. Then by the Lemma 6 H/K is isomorphic to additive group$W$. Thus any character of H/K is determined by its values on $\{(w,0) \mid w \in W\}$.Thus for each $w \in W$ define $\widetilde{\rho}_{(w,\chi)} : H/K \to \mathbb{C}^{\times}$ by*

$$\widetilde{\rho}_{(w,\chi)}\left(\widetilde{(w',0)}\right) = \chi(\langle w, w'\rangle).$$

$\widetilde{\rho}_{(w,\chi)}$ is a character of H/K. Now define $\rho_{(w,\chi)} : H \to \mathbb{C}^{\times}$ by $\rho_{(w,\chi)}(w',a) = \widetilde{\rho}_{(w,\chi)}\left(\widetilde{(w',0)}\right)$. $\rho_{(w,\chi)}$ is a character of H whose restriction to K is trivial.

Proof. The same proof as in the Theorem 2 works here. □

Corollary 13. *Theorem 5 determines q^{2n} characters of H.*

Proof. This result follows from Lemmas 7 and 8. □

Theorem 6. *Any representation of H when $p = 2$ is either one of the characters defined in the Therorem 4 or one of the characters defined in the Theorem 5.*

Proof. Let ψ be a character of H. If the restriction of ψ to $\{0\} \times F$ is a nontrivial character, then it is one of the characters in the Theorem 4. If the restriction of ψ to $\{0\} \times F$ is trivial, then it is one of the characters in the Theorem 5. Moreover, the number of characters that are built in the Theorems 4 and 5 are:

$$(q-1)\,q^{2n} + q^{2n} = q^{2n+1}.$$

which is the same as the order of H . □

References

[1] C. Curtis, & I. Reiner, *Representation Theory and Associative Algebra,* Wiley, New York (1988).

[2] W. Fulton and J. Harris, Representation Theory, A first Course, Springer-Verlag, New York (1991).

[3] T. Hungerford, *Algebra,* 5^{th} Ed. Springer-Verlag, New York (1989).

[4] Manouchehr Misaghian, "ON THE SMOOTH CHARACTERS OF THE ADDITIVE GROUP OF A LOCAL p-FIELD", preprint

[5] C. Moeglin, M.-F. Vigneras, And J.-L. Waldspurge, "*Correspondances de How sur un corps $p-adique$*," Lecture Notes in Math., Vol. 1291, Springer-Verlag, Berlin/New York, 1987.

[6] Brooks Roberts, "The Weil Representation and dual pairs", Lecture notes, Lecture I, unpublished.

[7] J-P. Serre, *Linear Representations of Finite Groups,* Springer-Verlag, New York, (1977).

[8] J.P. Serre, *A course in Arithmetic*, 5th edition, Springer-Verlag, New York, (1996)

[9] A.Weil, *Basic Number Theory*, Springer-Verlag, New York, (1967).

Department of Mathematics, Prairie view A & M University, Prairie View, TX 77446

E-mail address: `mamisaghian@pvamu.edu`