4.2 Zero of Polynomial Functions

Factor Theorem
Rational Zeros Theorem
Number of Zeros
Conjugate Zeros Theorem
Finding Zeros of a Polynomial Function
Factor Theorem

The polynomial $x - k$ is a factor of the polynomial $f(x)$ if and only if $f(k) = 0$. 
Example 2

FACTORING A POLYNOMIAL GIVEN A ZERO

Factor the following into linear factors if $-3$ is a zero of $f$.  

$$f(x) = 6x^3 + 19x^2 + 2x - 3$$

Solution  Since $-3$ is a zero of $f$, 

$$x - (-3) = x + 3$$  is a factor.

Use synthetic division to divide $f(x)$ by $x + 3$.

$\begin{array}{c|cccc}
-3 & 6 & 19 & 2 & -3 \\
 & & -18 & -3 & 3 \\
\hline
 & 6 & 1 & -1 & 0 \\
\end{array}$

The quotient is $6x^2 + x - 1$.  

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Example 2 FACTORING A POLYNOMIAL GIVEN A ZERO

Factor the following into linear factors if \(-3\) is a zero of \(f\). \(f(x) = 6x^3 + 19x^2 + 2x - 3\)

Solution \(x - (-3) = x + 3\) is a factor.

The quotient is \(6x^2 + x - 1\), so
\[
f(x) = (x + 3)(6x^2 + x - 1)
\]

\[
f(x) = (x + 3)(2x + 1)(3x - 1).
\]

Factor \(6x^2 + x - 1\).

These factors are all linear.
Rational Zeros Theorem

If \( \frac{p}{q} \) is a rational number written in lowest terms, and if \( \frac{p}{q} \) is a zero of \( f \), a polynomial function with integer coefficients, then \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient.
Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by \( f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 \).

a. List all possible rational zeros.

Solution For a rational number \( \frac{p}{q} \) to be zero, \( p \) must be a factor of \( a_0 = 2 \) and \( q \) must be a factor of \( a_4 = 6 \). Thus, \( p \) can be \( \pm 1 \) or \( \pm 2 \), and \( q \) can be \( \pm 1 \), \( \pm 2 \), \( \pm 3 \), or \( \pm 6 \). The possible rational zeros, \( \frac{p}{q} \) are,

\[
\pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}.
\]
Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by \( f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 \).

b. Find all rational zeros and factor \( f(x) \) into linear factors.

Solution Use the remainder theorem to show that 1 is a zero.

Use “trial and error” to find zeros.

\[
\begin{array}{cccccc}
1 & | & 6 & 7 & -12 & -3 & 2 \\
   & & 6 & 13 & 1 & -2 \\
\end{array}
\]

The 0 remainder shows that 1 is a zero. The quotient is \( 6x^3 + 13x^2 + x - 4 \), so \( f(x) = (x - 1)(6x^3 + 13x^2 + x - 2) \).
Example 3

USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by $f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$.

b. Find all rational zeros and factor $f(x)$ into linear equations.

Solution  Now, use the quotient polynomial and synthetic division to find that $-2$ is a zero.

$$
\begin{array}{c|cccc}
-2 & 6 & 13 & 1 & -2 \\
\hline
 & -12 & -2 & 2 \\
\end{array}
$$

$61 -1 \quad 0$  \[f(-2) = 0\]

The new quotient polynomial is $6x^2 + x - 1$. Therefore, $f(x)$ can now be factored.
Example 3  USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by \( f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 \).

b. Find all rational zeros and factor \( f(x) \) into linear equations.

Solution

\[
\begin{align*}
f(x) &= (x - 1)(x + 2)(6x^2 + x - 1) \\
&= (x - 1)(x + 2)(3x - 1)(2x + 1).
\end{align*}
\]
Example 3 USING THE RATIONAL ZERO THEOREM

Do the following for the polynomial function defined by \( f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 \).

b. Find all rational zeros and factor \( f(x) \) into linear equations.

Solution Setting \( 3x - 1 = 0 \) and \( 2x + 1 = 0 \) yields the zeros \( \frac{1}{3} \) and \( -\frac{1}{2} \). In summary the rational zeros are 1, \( -2, \frac{1}{3}, -\frac{1}{2} \), and the linear factorization of \( f(x) \) is

\[
f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2 = (x - 1)(x + 2)(3x - 1)(2x + 1).
\]

Check by multiplying these factors.
Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function \( f \) defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of \(-1, 2, \) and \(4; \) \( f(1) = 3 \)

Solution  These three zeros give \( x - (-1) = x + 1, \) \( x - 2, \) and \( x - 4 \) as factors of \( f(x). \)
Since \( f(x) \) is to be of degree 3, these are the only possible factors by the number of zeros theorem. Therefore, \( f(x) \) has the form

\[
f(x) = a(x + 1)(x - 2)(x - 4)
\]

for some real number \( a. \)
Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function \( f \) defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of \(-1, 2, \) and \(4; \) \( f(1) = 3 \)

Solution  To find \( a \), use the fact that \( f(1) = 3 \).

\[
f(1) = a(1 + 1)(1 - 2)(1 - 4)
\]

Let \( x = 1 \).

\[
3 = a(2)(-1)(-3)
\]

\[
3 = 6a
\]

Solve for \( a \).

\[
a = \frac{1}{2}
\]
Example 4

**FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)**

Find a function \( f \) defined by a polynomial of degree 3 that satisfies the given conditions.

a. Zeros of \(-1, 2, \text{ and } 4\); \( f(1) = 3 \)

**Solution**

Thus,

\[
f(x) = \frac{1}{2} (x + 1)(x - 2)(x - 4),
\]

or

\[
f(x) = \frac{1}{2} x^3 - \frac{5}{2} x^2 + x + 4. \quad \text{Multiply.}
\]
Find a function $f$ defined by a polynomial of degree 3 that satisfies the given conditions.

b. $-2$ is a zero of multiplicity 3; $f(-1) = 4$

Solution The polynomial function defined by $f(x)$ has the form

$$f(x) = a(x + 2)(x + 2)(x + 2)$$

$$= a(x + 2)^3.$$
Example 4

FINDING A POLYNOMIAL FUNCTION THAT SATISFIES GIVEN CONDITIONS (REAL ZEROS)

Find a function \( f \) defined by a polynomial of degree 3 that satisfies the given conditions.

b. \(-2\) is a zero of multiplicity 3; \( f(-1) = 4 \)

Solution  Since \( f(-1) = 4 \),

\[
f(-1) = a(-1+2)^3
\]

Remember:
\((x + 2)^3 \neq x^3 + 2^3\)

\[
4 = a(1)^3
\]

\[
a = 4,
\]

and \( f(x) = 4(x + 2)^3 = 4x^3 + 24x^2 + 48x + 32 \).