2.7 Graphing Techniques

- Stretching and Shrinking
- Reflecting
- Symmetry
- Even and Odd Functions
- Translations
Example 1

STRETCHING OR SHRINKING A GRAPH

Graph the function

a. \( g(x) = 2|x| \)

**Solution** Comparing the table of values for \( f(x) = |x| \) and \( g(x) = 2|x| \), we see that for corresponding \( x \)-values, the \( y \)-values of \( g \) are each twice those of \( f \). So the graph of \( g(x) \) is narrower than that of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
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<tr>
<td>0</td>
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<td>1</td>
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<tr>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>
Example 1

STRETCHING OR SHRINKING A GRAPH

Graph the function

a. \( g(x) = 2| x | \)

| \( x \) | \( f(x) = | x | \) | \( g(x) = 2| x | \) |
|-------|----------------|----------------|
| -2    | 2              | 4              |
| -1    | 1              | 2              |
| 0     | 0              | 0              |
| 1     | 1              | 2              |
| 2     | 2              | 4              |
Example 1

STRETCHING OR SHRINKING A GRAPH

b. \( h(x) = \frac{1}{2} |x| \)

Solution  The graph of \( h(x) \) is also the same general shape as that of \( f(x) \), but here the coefficient \( \frac{1}{2} \) causes the graph of \( h(x) \) to be wider than the graph of \( f(x) \), as we see by comparing the tables of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(</td>
<td>x</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 1

STRETCHING OR SHRINKING A GRAPH

b. \( h(x) = \frac{1}{2} \left| x \right| \)

| \( x \) | \( \left| x \right| \) | \( \frac{1}{2} \left| x \right| \) |
|-------|--------|--------|
| -2    | 2      | 1      |
| -1    | 1      | \( \frac{1}{2} \) |
| 0     | 0      | 0      |
| 1     | 1      | \( \frac{1}{2} \) |
| 2     | 2      | 1      |
Example 1  

STRETCHING OR SHRINKING A GRAPH

c. \( k(x) = |2x| \)

Solution  
Use Property 2 of absolute value \((ab = a|b|)\)

to rewrite \(2x\).

\[
k(x) = |2x| = 2|x| = 2x
\]

Property 2

The graph of \(k(x) = 2\) is the same as the graph of \(k(x) = 2|x|\) in part a.
Vertical Stretching or Shrinking of the Graph of a Function

Suppose that $a > 0$. If a point $(x, y)$ lies on the graph of $y = f(x)$, then the point $(x, ay)$ lies on the graph of $y = af(x)$.

a. If $a > 1$, then the graph of $y = af(x)$ is a **vertical stretching** of the graph of $y = f(x)$.

b. If $0 < a < 1$, then the graph of $y = af(x)$ is a **vertical shrinking** of the graph of $y = f(x)$.
Reflecting

Forming a mirror image of a graph across a line is called **reflecting the graph across the line**.
Example 2  REFLECTING A GRAPH ACROSS AN AXIS

Graph the function.

a. \( g(x) = -\sqrt{x} \)

**Solution** Every y-value of the graph is the negative of the corresponding y-value of \( f(x) = \sqrt{x} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} )</th>
<th>( g(x) = -\sqrt{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>-1</td>
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<tr>
<td>4</td>
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Example 2

REFLECTING A GRAPH ACROSS AN AXIS

Graph the function.

a. \( g(x) = -\sqrt{x} \)

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<tr>
<th>x</th>
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Example 2

REFLECTING A GRAPH ACROSS AN AXIS

Graph the function.

b. \( h(x) = \sqrt{-x} \)

Solution If we choose \( x \)-values for \( h(x) \) that are the negatives of those we use for \( f(x) \), the corresponding \( y \)-values are the same. The graph \( h \) is a reflection of the graph \( f \) across the \( y \)-axis.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = \sqrt{x} )</th>
<th>( h(x) = \sqrt{-x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>undefined</td>
<td>2</td>
</tr>
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<td>undefined</td>
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Graph the function.

b. \( h(x) = \sqrt{-x} \)

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<tr>
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</tr>
<tr>
<td>1</td>
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<td>undefined</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>undefined</td>
</tr>
</tbody>
</table>
Reflecting Across an Axis

The graph of \( y = -f(x) \) is the same as the graph of \( y = f(x) \) reflected across the x-axis. (If a point \((x, y)\) lies on the graph of \( y = f(x) \), then \((x, -y)\) lies on this reflection.

The graph of \( y = f(-x) \) is the same as the graph of \( y = f(x) \) reflected across the y-axis. (If a point \((x, y)\) lies on the graph of \( y = f(x) \), then \((-x, y)\) lies on this reflection.)
Symmetry with Respect to An Axis

The graph of an equation is symmetric with respect to the $y$-axis if the replacement of $x$ with $-x$ results in an equivalent equation. The graph of an equation is symmetric with respect to the $x$-axis if the replacement of $y$ with $-y$ results in an equivalent equation.
Example 3

TESTING FOR SYMMETRY WITH RESPECT TO AN AXIS

Test for symmetry with respect to the $x$-axis and the $y$-axis.

a. $y = x^2 + 4$

**Solution** Replace $x$ with $-x$.

- $y = x^2 + 4$
- $y = (-x)^2 + 4$
- $y = x^2 + 4$

Equivalent
Example 3  TESTING FOR SYMMETRY WITH RESPECT TO AN AXIS

Test for symmetry with respect to the $x$-axis and the $y$-axis.

a. $y = x^2 + 4$

Solution  The result is the same as the original equation and the graph is symmetric with respect to the $y$-axis.

\[ y = x^2 + 4 \]

\[ y = (-x)^2 + 4 \] Equivalent

\[ y = x^2 + 4 \]
Example 3  TESTING FOR SYMMETRY WITH RESPECT TO AN AXIS

Test for symmetry with respect to the $x$-axis and the $y$-axis.

b. $x = y^2 - 3$

Solution  Replace $y$ with $-y$.

\[
x = y^2 - 3
\]

\[
x = (-y)^2 - 3
\]

\[
= y^2 - 3
\]

Equivalent
Example 3

TESTING FOR SYMMETRY WITH RESPECT TO AN AXIS

Test for symmetry with respect to the $x$-axis and the $y$-axis.

b. $x = y^2 - 3$

**Solution** The graph is symmetric with respect to the $x$-axis.

$x = y^2 - 3$

$x = (-y)^2 - 3$

$= y^2 - 3$

Equivalent
Test for symmetry with respect to the $x$-axis and the $y$-axis.

c. $x^2 + y^2 = 16$

**Solution** Substituting $-x$ for $x$ and $-y$ in for $y$, we get $(-x)^2 + y^2 = 16$ and $x^2 + (-y)^2 = 16$. Both simplify to $x^2 + y^2 = 16$. The graph, a circle of radius 4 centered at the origin, is symmetric with respect to both axes.
TESTING FOR SYMMETRY WITH RESPECT TO AN AXIS

Example 3

Test for symmetry with respect to the $x$-axis and the $y$-axis.

d. $2x + y = 4$

**Solution** In $2x + y = 4$, replace $x$ with $-x$ to get $-2x + y = 4$. Replace $y$ with $-y$ to get $2x - y = 4$. Neither case produces an equivalent equation, so this graph is not symmetric with respect to either axis.
Symmetry with Respect to the Origin

The graph of an equation is symmetric with respect to the origin if the replacement of both $x$ with $-x$ and $y$ with $-y$ results in an equivalent equation.
Example 4

TESTING FOR SYMMETRY WITH RESPECT TO THE ORIGIN

Is this graph symmetric with respect to the origin?

a. $x^2 + y^2 = 16$

Solution

Replace $x$ with $-x$ and $y$ with $-y$.

$x^2 + y^2 = 16$

$(-x)^2 + (-y)^2 = 16$

$x^2 + y^2 = 16$

The graph is symmetric with respect to the origin.
Example 4

TESTING FOR SYMMETRY WITH RESPECT TO THE ORIGIN

Is this graph symmetric with respect to the origin?

b. \( y = x^3 \)

Solution

Replace \( x \) with \(-x\) and \( y \) with \(-y\).

\[
\begin{align*}
\text{Original} & : & y &= x^3 \\
\text{After transformation} & : & -y &= (-x)^3 \\
\text{Simplified} & : & -y &= -x^3 \\
\text{Conclusion} & : & y &= x^3
\end{align*}
\]

The graph is symmetric with respect to the origin.
### Symmetry with Respect to:

<table>
<thead>
<tr>
<th></th>
<th>x-axis</th>
<th>y-axis</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation is</td>
<td>y is replaced with (-y)</td>
<td>x is replaced with (-x)</td>
<td>x is replaced with (-x) and y is replaced with (-y)</td>
</tr>
<tr>
<td>unchanged if:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Example</td>
<td><img src="" alt="Graph" /></td>
<td><img src="" alt="Graph" /></td>
<td><img src="" alt="Graph" /></td>
</tr>
</tbody>
</table>

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**Example:**

- For a function to be symmetric with respect to the x-axis, replace every \(y\) with \(-y\).
- For symmetry with respect to the y-axis, replace every \(x\) with \(-x\).
- For symmetry with respect to the origin, replace every \(x\) with \(-x\) and every \(y\) with \(-y\).
Even and Odd Functions

A function $f$ is called an **even function** if $f(-x) = f(x)$ for all $x$ in the domain of $f$. (Its graph is symmetric with respect to the $y$-axis.)

A function $f$ is called an odd function if $f(-x) = -f(x)$ for all $x$ in the domain of $f$. (Its graph is symmetric with respect to the origin.)
Example 5  DETERMINING WHETHER FUNCTIONS ARE EVEN, ODD, OR NEITHER

Decide whether each function defined is 

**even**, **odd**, or **neither**.

a. \( f(x) = 8x^4 - 3x^2 \)

**Solution** Replacing \( x \) in \( f(x) = 8x^4 - 3x^2 \) with \(-x\) gives:

\[
f(-x) = 8(-x)^4 - 3(-x)^2 = 8x^4 - 3x^2 = f(x)
\]

Since \( f(-x) = f(x) \) for each \( x \) in the domain of the function, \( f \) is even.
Example 5

DETERMINING WHETHER FUNCTIONS ARE EVEN, ODD, OR NEITHER

Decide whether each function defined is even, odd, or neither.

b. \( f(x) = 6x^3 - 9x \)

Solution

\[
\begin{align*}
-f(x) &= -6x^3 + 9x \\
&= -f(x)
\end{align*}
\]

The function \( f \) is odd because \( f(-x) = -f(x) \).
Example 5

DETERMINING WHETHER FUNCTIONS ARE EVEN, ODD, OR NEITHER

Decide whether each function defined is even, odd, or neither.

c. \( f(x) = 3x^2 + 5x \)

Solution

\[
\begin{align*}
  f(x) &= 3x^2 + 5 \\
  f(-x) &= 3(-x)^2 + 5(-x) \\
         &= 3x^2 - 5x
\end{align*}
\]

Since \( f(-x) \neq f(x) \) and \( f(-x) \neq -f(x) \), \( f \) is neither even nor odd.
Example 6

TRANSLATING A GRAPH VERTICALLY

Graph \( g(x) = |x| - 4 \)

Solution

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>
Vertical Translations

Original graph

$y = f(x) + 2$
Vertical translation 2 units up

$y = f(x)$

$y = f(x) - 3$
Vertical translation 3 units down
Example 7

TRANSLATING A GRAPH HORIZONTALLY

Graph \( g(x) = |x - 4| \)

Solution

\[
\begin{array}{c|c|c}
 x & f(x) & g(x) \\
\hline
-2 & 2 & 6 \\
0 & 0 & 4 \\
2 & 2 & 2 \\
4 & 4 & 0 \\
6 & 6 & 2 \\
\end{array}
\]

\[
\begin{aligned}
& f(x) = |x| \\
& g(x) = |x - 4| \\
& (0, -4)
\end{aligned}
\]
Horizontal Translations

Original graph
\[ y = f(x) \]

Horizontal translation
2 units to the left
\[ y = f(x + 2) \]

Horizontal translation
3 units to the right
\[ y = f(x - 3) \]
Horizontal Translations

If a function $g$ is defined by $g(x) = f(x - c)$, where $c$ is a real number, then for every point $(x, y)$ on the graph of $f$, there will be a corresponding point $(x + c)$ on the graph of $g$. The graph of $g$ will be the same as the graph of $f$, but translated $c$ units to the right if $c$ is positive or $|c|$ units to the left if $c$ is negative. The graph is called a **horizontal translation** of the graph of $f$. 
Example 8  USING MORE THAN ONE TRANSAFORMATION ON GRAPHS

Graph the function.

a. $f(x) = -|x + 3| + 1$

Solution  To graph this, the lowest point on the graph of $y = |x|$ is translated 3 units to the left and up one unit. The graph opens down because of the negative sign in front of the absolute value expression, making the lowest point now the highest point on the graph. The graph is symmetric with respect to the line $x = -3$. 
Example 8

USING MORE THAN ONE TRANSFORMATION ON GRAPHS

Graph the function.

a. \( f(x) = -|x + 3| + 1 \)
Graph the function.

b. \( h(x) = |2x - 4| \)

**Solution** To determine the horizontal translation, factor out 2.

\[
h(x) = |2x - 4| = |2(x - 2)| \quad \text{Factor out 2.}
\]

\[
= 2|x - 2| \quad \text{ab} = |a|\cdot|b|
\]

\[
= 2|x - 2| \quad |2| = 2
\]
Example 8

USING MORE THAN ONE TRANSFORMATION ON GRAPHS

Graph the function.

b. \( h(x) = -|2x - 4| \)

Solution

Factor out 2.

\[
h(x) = |2x - 4| = 2|x - 2|
\]
Example 8

USING MORE THAN ONE TRANSFORMATION ON GRAPHS

Graph the function.

c. \( g(x) = -\frac{1}{2}x^2 + 4 \)

**Solution**  It will have the same shape as that of \( y = x^2 \), but is wider (that is, shrunken vertically) and reflected across the x-axis because of the negative coefficient and then translated 4 units up.
Example 8

USING MORE THAN ONE TRANSFORMATION ON GRAPHS

Graph the function.

c. \( g(x) = -\frac{1}{2}x^2 + 4 \)

Solution
Summary of Graphing Techniques

In the descriptions that follow, assume that $a > 0$, $h > 0$, and $k > 0$. In comparison with the graph of $y = f(x)$:

1. The graph of $y = f(x) + k$ is translated $k$ units up.
2. The graph of $y = f(x) - k$ is translated $k$ units down.
3. The graph of $y = f(x + h)$ is translated $h$ units to the left.
4. The graph of $y = f(x - h)$ is translated $h$ units to the right.
5. The graph of $y = af(x)$ is a vertical stretching of the graph of $y = f(x)$ if $a > 1$. It is a vertical shrinking if $0 < a < 1$.
6. The graph of $y = af(x)$ is a horizontal stretching of the graph of $y = f(x)$ if $0 < a < 1$. It is a horizontal shrinking if $a > 1$.
7. The graph of $y = -f(x)$ is reflected across the x-axis.
8. The graph of $y = f(-x)$ is reflected across the y-axis.