2.8 Function Operations and Composition

Arithmetic Operations on Functions
The Difference Quotient
Composition of Functions and Domain
Operations of Functions

Given two functions $f$ and $g$, then for all values of $x$ for which both $f(x)$ and $g(x)$ are defined, the functions $f + g$, $f - g$, $fg$, and $f/g$ are defined as follows.

\[
(f + g)(x) = f(x) + g(x) \quad \text{Sum}
\]
\[
(f - g)(x) = f(x) - g(x) \quad \text{Difference}
\]
\[
(fg)(x) = f(x)\,g(x) \quad \text{Product}
\]
\[
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0 \quad \text{Quotient}
\]
Example 1

USING OPERATIONS ON FUNCTIONS

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find the following.

a. $(f + g)(1)$

Solution Since $f(1) = 2$ and $g(1) = 8$, use the definition to get

$$(f + g)(1) = f(1) + g(1) = 2 + 8 = 10$$
Example 1 USING OPERATIONS ON FUNCTIONS

Let $f(x) = x^2 + 1$ and $g(x) = 3x + 5$. Find the following.

b. $(f - g)(-3)$

**Solution** Since $f(-3) = 10$ and $g(-3) = -4$, use the definition to get

$$ (f - g)(-3) = f(-3) - g(-3) = 10 - (-4) = 14 $$
Example 1

USING OPERATIONS ON FUNCTIONS

Let \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 5 \). Find the following.

\[ c. \ (fg)(5) \]

**Solution**  Since \( f(5) = 26 \) and \( g(5) = 20 \), use the definition to get

\[
(fg)(5) = f(5) \cdot g(5)
\]

\[ = 26 \cdot 20 \]

\[ = 520 \]
Example 1

USING OPERATIONS ON FUNCTIONS

Let \( f(x) = x^2 + 1 \) and \( g(x) = 3x + 5 \). Find the following.

d. \( \left( \frac{f}{g} \right)(0) \)

**Solution** Since \( f(0) = 1 \) and \( g(0) = 5 \), use the definition to get

\[
\left( \frac{f}{g} \right)(0) = \frac{f(0)}{g(0)} = \frac{1}{5}
\]
Example 2 USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let 
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1} \]. Find the following.

a. \((f + g)(x)\)

Solution

\[ (f + g)(x) = f(x) + g(x) = 8x - 9 + \sqrt{2x - 1} \]
Example 2  

**USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS**

Let 
\[ f(x) = 8x - 9 \]  
and  
\[ g(x) = \sqrt{2x - 1}. \]  
Find the following.

b. \((f - g)(x)\)

**Solution**

\[ (f - g)(x) = f(x) - g(x) = 8x - 9 - \sqrt{2x - 1} \]
Example 2

USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1}. \] Find the following.

c. \((fg)(x)\)

Solution

\[ (fg)(x) = f(x) \cdot g(x) = (8x - 9) \sqrt{2x - 1} \]
Example 2

USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1} \]. Find the following.

d. \[ \left( \frac{f}{g} \right)(x) \]

Solution

\[ \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} = \frac{8x - 9}{\sqrt{2x - 1}} \]
Example 2 

USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1}. \] Find the following.

e. Give the domains of the functions.

Solution To find the domains of the functions, we first find the domains of \( f \) and \( g \).

The domain of \( f \) is the set of all real numbers \((-\infty, \infty)\).
Example 2

Using Operations of Functions and Determining Domains

Let

\[ f(x) = 8x - 9 \quad \text{and} \quad g(x) = \sqrt{2x - 1}. \]

Find the following.

e. Give the domains of the functions.

Solution  Since \( g(x) = \sqrt{2x - 1} \), the domain of \( g \) includes just the real numbers that make \( 2x - 1 \) nonnegative. Solve \( 2x - 1 \geq 0 \) to get \( x \geq \frac{1}{2} \). The domain of \( g \) is \( \left[ \frac{1}{2}, \infty \right) \).
Example 2

USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1} \]. Find the following.

e. Give the domains of the functions.

Solution

The domains of \( f + g \), \( f - g \), \( fg \) are the intersection of the domains of \( f \) and \( g \), which is

\[ (-\infty, \infty) \cap \left[ \frac{1}{2}, \infty \right) = \left[ \frac{1}{2}, \infty \right) \]
Example 2

USING OPERATIONS OF FUNCTIONS AND DETERMINING DOMAINS

Let
\[ f(x) = 8x - 9 \] and \[ g(x) = \sqrt{2x - 1}. \] Find the following.

e. Give the domains of the functions.

Solution  The domains of \( \frac{f}{g} \) includes those real numbers in the intersection for which
\[ g(x) = \sqrt{2x - 1} \neq 0; \]
that is, the domain of \( \frac{f}{g} \) is \( \left( \frac{1}{2}, \infty \right) \).
If possible, use the given representations of functions $f$ and $g$ to evaluate …

$$(f + g)(4), \quad (f - g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0).$$
Example 3

EVALUATING COMBINATIONS OF FUNCTIONS

\[(f + g)(4), \quad (f - g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0)\].

\[
\begin{align*}
(f + g)(4) &= f(4) + g(4) \\
&= 9 + 2 \\
&= 11
\end{align*}
\]

For \((f - g)(-2)\), although \(f(-2) = -3\), \(g(-2)\) is undefined because \(-2\) is not in the domain of \(g\).
Example 3

\[ (f + g)(4), \quad (f - g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0). \]

**EVALUATING COMBINATIONS OF FUNCTIONS**

\[ y = f(x) \quad y = g(x) \]

\[ f(4) = 9 \quad g(4) = 2 \]

\[ = f(4) + g(4) \]

\[ = 9 + 2 = 11 \]

The domains of \( f \) and \( g \) include 1, so

\[ (fg)(1) = f(1) \cdot g(1) = 3 \cdot 1 = 3 \]
Example 3

EVALUATING COMBINATIONS OF FUNCTIONS

\((f + g)(4),\ (f - g)(-2),\ (fg)(1),\ \text{and}\ \left(\frac{f}{g}\right)(0)\).

\[
\begin{align*}
(f + g)(4) &= 9 + 2 = 11 \\
g(4) &= 2 \\
f(4) &= 9 \\
\end{align*}
\]

The graph of \(g\) includes the origin, so \(g(0) = 0\).

Thus, \(\left(\frac{f}{g}\right)(0)\) is undefined.
Example 3

EVALUATING COMBINATIONS OF FUNCTIONS

If possible, use the given representations of functions \( f \) and \( g \) to evaluate

\[
(f + g)(4), \quad (f - g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0).
\]

b. \[
\begin{array}{c|c|c}
 x & f(x) & g(x) \\
 \hline
 -2 & -3 & \text{undefined} \\
 0 & 1 & 0 \\
 1 & 3 & 1 \\
 1 & 1 & \text{undefined} \\
 4 & 9 & 2 \\
\end{array}
\]

\[
f(4) = 9 \quad g(4) = 2
\]

\[
= f(4) + g(4)
\]

\[
= 9 + 2 = 11
\]

In the table, \( g(-2) \) is undefined.

Thus, \( (f-g)(-2) \) is undefined.
### Example 3

If possible, use the given representations of functions $f$ and $g$ to evaluate

$$(f + g)(4),\quad (f - g)(-2),\quad (fg)(1),\quad \text{and} \quad \left(\frac{f}{g}\right)(0).$$

#### b.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$h(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$-3$</td>
<td>undefined</td>
</tr>
<tr>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$1$</td>
<td>$3$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>undefined</td>
</tr>
<tr>
<td>$4$</td>
<td>$9$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

- $f(4) = 9, \quad g(4) = 2$
- $= f(4) + g(4)$
  
  $= 9 + 2 = 11$

- $(fg)(1) = f(1) \cdot g(1) = 3 \cdot 1 = 3$
Example 3

EVALUATING COMBINATIONS OF FUNCTIONS

If possible, use the given representations of functions $f$ and $g$ to evaluate

$$(f + g)(4), \quad (f - g)(-2), \quad (fg)(1), \quad \text{and} \quad \left(\frac{f}{g}\right)(0).$$

b. $f(4) = 9 \quad g(4) = 2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$h(x)$</th>
<th>$\frac{f}{g}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-3</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>undefined</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

$= f(4) + g(4)$

$= 9 + 2 = 11$

And $\frac{f}{g}(0) = \frac{f(0)}{g(0)}$ is undefined since $g(0) = 0$
Example 3

EVALUATING COMBINATIONS OF FUNCTIONS

If possible, use the given representations of functions $f$ and $g$ to evaluate $(f + g)(4)$, $(f - g)(-2)$, $(fg)(1)$, and $\left(\frac{f}{g}\right)(0)$.

c. $f(x) = 2x + 1$, $g(x) = \sqrt{x}$

$(f + g)(4) = f(4) + g(4) = (2 \cdot 4 + 1) + \sqrt{4} = 9 + 2 = 11$

$(f - g)(-2) = f(-2) + g(-2) = [2(-2) + 1] - \sqrt{-2}$

is undefined.

$(fg)(1) = f(1) \cdot g(1) = (2 \cdot 1 + 1) \sqrt{1} = 3(1) = 3$
EVALUATING COMBINATIONS OF FUNCTIONS

Example 3

c. \( f(x) = 2x + 1 \), \( g(x) = \sqrt{x} \)

\((f + g)(4) = f(4) + g(4) = (2 \cdot 4 + 1) + \sqrt{4} = 9 + 2 = 11\)

\((f - g)(-2) = f(-2) + g(-2) = [2(-2) + 1] - \sqrt{-2}\)

is undefined.

\((fg)(1) = f(1) \cdot g(1) = (2 \cdot 1 + 1) \sqrt{1} = 3(1) = 3\)

\(\left(\frac{f}{g}\right)\) is undefined.
Example 4  FINDING THE DIFFERENCE QUOTIENT

Let \( f(x) = 2x^2 - 3x \). Find the difference quotient and simplify the expression.

Solution

Step 1  Find the first term in the numerator, \( f(x + h) \). Replace the \( x \) in \( f(x) \) with \( x + h \).

\[
f(x + h) = 2(x + h)^2 - 3(x + h)
\]
Example 4  FINDING THE DIFFERENCE QUOTIENT

Let \( f(x) = 2x - 3x \). Find the difference quotient and simplify the expression.

**Solution**

**Step 2** Find the entire numerator \( f(x + h) - f(x) \).

\[
\begin{align*}
f(x + h) - f(x) &= \left[ 2(x + h)^2 - 3(x + h) \right] - (2x^2 - 3x) \\
&= 2(x^2 + 2xh + h^2) - 3(x + h) - (2x^2 - 3x)\end{align*}
\]

Remember this term when squaring \( x + h \)
Example 4  FINDING THE DIFFERENCE QUOTIENT

Let \( f(x) = 2x - 3x \). Find the difference quotient and simplify the expression.

Solution

Step 2  Find the entire numerator \( f(x + h) - f(x) \).

\[
= 2(x^2 + 2xh + h^2) - 3(x + h) - (2x^2 - 3x)
\]

\[
= 2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x
\]

Distributive property

\[
= 4xh + 2h^2 - 3h
\]

Combine terms.
Example 4  FINDING THE DIFFERENCE QUOTIENT

Let \( f(x) = 2x - 3x \). Find the difference quotient and simplify the expression.

Solution

**Step 3**  Find the quotient by dividing by \( h \).

\[
\frac{f(x + h) - f(x)}{h} = \frac{4xh + 2h^2 - 3h}{h}
\]

Substitute.

\[
= \frac{h(4x + 2h - 3)}{h}
\]

Factor out \( h \).

\[
= 4x + 2h - 3
\]

Divide.
Caution Notice that $f(x + h)$ is not the same as $f(x) + f(h)$. For $f(x) = 2x^2 - 3x$ in Example 4.

$$f(x + h) = 2(x + h)^2 - 3(x + h) = 2x^2 + 4xh + 2h^2 - 3x - 3h$$

but

$$f(x) + f(h) = (2x^2 - 3x) + (2h^2 - 3h) = 2x^2 - 3x + 2h^2 - 3h$$

These expressions differ by $4xh$. 
Composition of Functions and Domain

If $f$ and $g$ are functions, then the **composite function**, or **composition**, of $g$ and $f$ is defined by

$$(g \circ f)(x) = g(f(x)).$$

The **domain of** $g \circ f$ is the set of all numbers $x$ in the domain of $f$ such that $f(x)$ is in the domain of $g$. 

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Example 5

EVALUATING COMPOSITE FUNCTIONS

Let \( f(x) = 2x - 1 \) and \( g(x) = \frac{4}{x - 1} \)

a. Find \( (f \circ g)(2) \).

Solution  First find \( g(2) \). Since \( g(x) = \frac{4}{x - 1} \),

\[
g(2) = \frac{4}{2 - 1} = \frac{4}{1} = 4
\]

Now find \( (f \circ g)(2) = f(g(2)) = f(4) : \)

\[
f(g(2)) = f(4) = 2(4) - 1 = 7
\]
Example 5

EVALUATING COMPOSITE FUNCTIONS

Let \( f(x) = 2x - 1 \) and \( g(x) = \frac{4}{x - 1} \)

b. Find \((g \circ f)(-3)\).

Solution \((f \circ g)(-3) = g(f(-3)) = g(-7)\):

\[
= \frac{4}{-7 - 1} = \frac{4}{-8} = -\frac{1}{2}.
\]

Don’t confuse composition with multiplication
Example 8

SHOWING THAT $$(g \circ f)(x) \neq (f \circ g)(x)$$

Let $f(x) = 4x + 1$ and $g(x) = 2x^2 + 5x$.

Show that $(g \circ f)(x) \neq (g \circ f)(x)$ in general.

Solution

First, find $(g \circ f)(x)$.

$$(g \circ f)(x) = g(f(x)) = g(4x + 1) \quad f(x) = 4x + 1$$

$$= 2(4x + 1)^2 + 5(4x + 1) \quad g(x) = 2x^2 + 5x$$

Square $4x + 1$; distributive property.

$$= 2(16x^2 + 8x + 1) + 20x + 5$$
Example 8

SHOWING THAT \((g \circ f)(x) \neq (f \circ g)(x)\)

Let \(f(x) = 4x + 1\) and \(g(x) = 2x^2 + 5x\).
Show that \((g \circ f)(x) \neq (g \circ f)(x)\) in general.

Solution
First, find \((g \circ f)(x)\).

\[
(g \circ f)(x) = 2(16x^2 + 8x + 1) + 20x + 5
\]

Distributive property.

\[
= 32x^2 + 16x + 2 + 20x + 5
\]

Combine terms.

\[
= 32x^2 + 36x + 7
\]
Example 8

SHOWING THAT \((g \circ f)(x) \neq (f \circ g)(x)\)

Let \(f(x) = 4x + 1\) and \(g(x) = 2x^2 + 5x\).
Show that \((g \circ f)(x) \neq (g \circ f)(x)\) in general.

Solution

Now find \((f \circ g)(x)\).

\[
(f \circ g)(x) = f(g(x))
\]

\[
= f\left(2x^2 + 5x\right) \quad g(x) = 2x^2 + 5x
\]

\[
= 4\left(2x^2 + 5x\right) + 1 \quad f(x) = 4x + 1
\]

\[
= 8x^2 + 20x + 1 \quad \text{Distributive property}
\]

So... \((g \circ f)(x) \neq (f \circ g)(x)\).