SUMMARY
Phys 2513 (University Physics I)
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- **Position Vector** (m): $$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

- **Velocity** (m/s): $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

- **Linear Momentum** (kg·m/s): $$\mathbf{p} = m\mathbf{v}$$

- **Acceleration** (m/s^2): $$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$$

- **Centripetal Acceleration.** Acceleration is $\frac{d\mathbf{v}}{dt}$. At any instant, the motion of an object can be viewed as being tangent to a circle. The acceleration can represent either a change in speed of the object (tangential acceleration), a change in direction of the object (radial acceleration), or both. The radial acceleration is the **centripetal acceleration**. The magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r}$$

where r is the radius of that instantaneous circle. Its direction is toward the center of that instantaneous circle, $\mathbf{a}_c = -\frac{v^2}{r}\hat{\mathbf{r}} = -\frac{v^2\mathbf{r}}{r^2}$. (*Centripetal* means *center-seeking.*) Note that for straight-line motion, $r \to \infty$, and $a_c \to 0$. For *uniform circular motion* the tangential speed v is just the circumference of the circle divided by the period (T) or time to complete one revolution, i.e., $v = 2\pi/T = 2\pi f$, where $f = 1/T$ is the frequency.

- **Force** (N=kg·m/s^2): $$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$$

- **Impulse:** $$\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F}dt' = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$$

- **Work** (J=N·m): $$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r}$$

- **Power** (W=J/s): $$P = \mathbf{F}\cdot\mathbf{v}; \text{ generally, } P = \frac{dE}{dt}$$

- **Kinetic Energy** (J): $$K = \frac{1}{2}mv^2$$

• **Potential Energy** (J): If the work is independent of path, then the force is **conservative** and can be derived from a potential energy U, where

$$\mathbf{F} = -\nabla U$$

— Example: Gravitational potential, $U_g = mg(y - y_\circ)$, where y is the vertical direction.

— Example: Potential energy of a spring is $U_s = \frac{1}{2}kx^2$, where x is the displacement from its equilibrium position.

Non-conservative forces (e.g., friction) are those for which the work depends on the path.

• **Newton's First Law** states that in the absence of a net external force, when viewed from an inertial frame, $\mathbf{v}$ of an object is constant. An **inertial frame of reference** is one for which $\mathbf{a} = 0$.

• **Galilean Transformations.** Position, velocity and acceleration measured in frame S are related to those quantities measured in a frame S′, moving with constant velocity $\mathbf{v}_S$ relative to S, by

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}_S t; \;\; \mathbf{v}' = \mathbf{v} - \mathbf{v}_S; \;\; \mathbf{a}' = \mathbf{a}$$

Galilean transformation is only valid for kinematic situations in which velocities are much less than the speed of light ($c = 3 \times 10^8$ m/s). Galilean transformations are inaccurate for electromagnetic phenomena, quantum phenomena, and kinematic situations in which v/c is a significant fraction of unity.

• **Fictitious Forces.** Fictitious forces appear when observing from a non-inertial frame, i.e., observing from a frame undergoing acceleration. Examples are the centrifugal force (responsible for the sensation of gravity) and Coriolis force (which causes hurricanes and whirlpools).

• **Newton's Second Law** states that $\sum \mathbf{F} = m\mathbf{a}$, i.e., the acceleration of an object is proportional to the net force on the object and inversely proportional to the (inertial) mass. (Note that if $\sum \mathbf{F} = 0$, then $\mathbf{a}$=0, and the object is in **equilibrium**.)

• **Equation of Motion:**

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \frac{\sum \mathbf{F}}{m} = \frac{\mathbf{F}_{net}}{m}$$

— Example: If $\mathbf{a} = \textit{constant}$, then the solution is

$$\mathbf{r} = \mathbf{r}_o + \mathbf{v}_o t + \frac{1}{2}\mathbf{a}t^2; \;\; \mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{v}_o + \mathbf{a}t$$

Note that time can be eliminated from these equations to yield for each component:

$$v_i^2 = v_\circ^2 + 2a_i \Delta r_i, \text{ where } i = x, y, z.$$

— Example: Spring, $F = ma_x = -kx$ or pendulum, $F = -mgx/L$

$$x = x_{max} \cos\left(\sqrt{\frac{k}{m}}\, t\right)\ ;\ \ x = x_{max} \cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

A **free-body diagram** allow one to visualize the forces on an object with the help of Newton's laws. The equations of motion can then be used to describe the motions.

• **Frictional Forces.** Resistance to motion is offered on surfaces or in a viscous medium. On a surface, **static friction**, $\mathbf{f}_s$, opposes a force applied *parallel* to the surface, $\mathbf{F}_{\|}$, to try to keep the object motionless w.r.t. the surface. While motionless, $\mathbf{f}_s = -\mathbf{F}_{\|}$, and the magnitude of the friction force

$$f_s \leq \mu_s n$$

where μ_s is the coefficient of static friction, and n is the magnitude of the normal force applied by the surface on the object. If the static frictional force is exceeded, the object will accelerate. If the object slips, the frictional force is reduced to that of **kinetic friction**

$$\mathbf{f}_k = -\mu_k n \hat{\mathbf{v}} = -\mu_k n \left(\frac{\mathbf{v}}{v}\right)$$

When an object moves through a viscous media, it collides with the individual constituents of that medium. Like a fly hitting the windshield of a car, the object applies an impulse, i.e., a force on the individual constituents, and there is a reactive force (by Newton's third law) on the object from each of the constituents. We expect a **resistive force**, $\mathbf{R} = -b\mathbf{v}$. An object falling through the medium feels a net force

$$m\frac{dv}{dt} = g - bv$$

Solving for its speed

$$v = v_T\left(1 - e^{-t/\tau}\right)$$

where $v_T = \frac{mg}{b}$ is the *terminal speed* of the object.

In the case of an object traveling at high speeds in air, the magnitude of the *air drag* is $R = \frac{1}{2}D\rho A v^2$, where ρ is the density of air, A is the cross-sectional area of the object, and D is the *drag coefficient*. Thje object will feel an acceleration $a = g - \left(\frac{D\rho A}{2m}\right) v^2$ until it reaches its terminal velocity $v_T = \left(\frac{2mg}{D\rho A}\right)^{1/2}$.

• **Newton's Third Law** states that for every action there is an equal and opposite reaction, i.e., $\mathbf{F}_{12} = -\mathbf{F}_{21}$. An isolated force cannot exist in nature.

• **Newton's law of universal gravitation** states that every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Specifically, the gravitational force exerted by particle 1 on particle 2 is

$$\mathbf{F}_{12}(r) = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}_{12}$$

where r is the distance between them, and the *universal gravitational constant* $G = 6.673 \times 10^{-11}$ N·m^2/kg^2. The gravitational potential energy associated with a pair of particles separated by a distance r is

$$U_g(r) = -\frac{Gm_1 m_2}{r}$$

Note that gravitation is a conservative force, and $\mathbf{F}_g = -\nabla U_g$. The gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center. For example, the force exerted by the Earth on a particle of mass m is

$$\mathbf{F}_g(r) = -G\frac{M_E m}{r^2}\hat{\mathbf{r}}$$

where M_E is the Earth's mass and $\mathbf{r}$ is measured from Earth's center. The gravitational potential energy is

$$U_g(r) = -G\frac{M_E m}{r}$$

• **Kepler's laws:**

1. All planets move in elliptical orbits with the Sun at one focus.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

• **Conservation of Energy** states that energy can neither be created or destroyed – energy is always conserved. If the total energy of a system E_{system} changes, it can only be due to the fact that energy has crossed the system boundary by a transfer mechanism T

$$\Delta E_{system} = \sum T$$

E_{system} is the total energy of the system, including $E_{mechanical} = K + U$ and $E_{internal}$, which includes several kinetic and potential energies internal to the system, such as the rest-mass energy, chemical energy, vibrational energy, etc., that we have not addressed yet – except for *friction* which changes mechanical energy into internal energy. T represents mechanisms by which energy is transferred across the system boundary, including heat, work, electromagnetic radiation, etc.

Energy can neither be created nor destroyed, so conservation of energy is a "bookkeeping" problem. Think of energy as currency and "follow the money". *Work transforms energy from one form to another.* At this point, we have discussed only kinetic energy, K, potential energy P, the work done by friction, $-f_k d$ (at this point, friction is the only non-conservative force discussed), and the work done by other stated forces, $\sum W_{otherforces}$. These yield the following conservation of energy relation

$$K_f + P_f = K_o + P_o - f_k d + \sum W_{otherforces}$$

• **Conservation of Momentum.** A direct consequence of Newton's third law is that the total momentum of an isolated system (i.e., a system subject to no external forces) at all times equals its initial momentum

$$\sum_i m_i \mathbf{v}_{if} = \sum_i m_i \mathbf{v}_{io}$$

An **elastic** collision is one in which the kinetic energy is conserved

$$\sum_i \frac{1}{2} m_i v_{if}^2 = \sum_i \frac{1}{2} m_i v_{io}^2 \quad \text{(elastic collisions)}$$

Extended Mass Distributions

The discussion so far has dealt with the motion of a point particle or mass. Everything discussed to here is valid for an extended mass distribution as well. *The motion of an extended object can be described as the motion of the center of mass (CM) plus the motion about the center of mass.* Newton's law still apply to all particles of the system or extended body. Forces may be external or internal to the system. However, because of Newton's third law all *internal forces* occur in equal and opposite pairs and hence *do not contribute to the motion of the center of mass.*

Motion of the Center of Mass

If M is the total mass of the extended object, then

- **Position Vector of CM** (m):

$$\mathbf{r}_{\rm cm} = \frac{1}{M}\sum_i m_i \mathbf{r}_i = \frac{1}{M}\int \mathbf{r}\,dm = \frac{1}{M}\int \rho \mathbf{r}\,dV$$

where ρ is the mass density and dV is the volume element. $dV = dxdydz$ (rectangular), $dV = rdrd\phi dz$ (cylindrical) ($\int d\phi dz = 2\pi L$), $dV = r^2 \sin\theta d\theta d\phi$ (spherical) ($\int \sin\theta d\theta d\phi = 4\pi$).

- **Velocity of CM** (m/s): $$\mathbf{v}_{\rm cm} = \dot{\mathbf{r}}_{\rm cm} = \frac{d\mathbf{r}_{\rm cm}}{dt}$$

- **Total Linear Momentum of Body is** (kg·m/s):

$$\mathbf{p_{tot}} = M\mathbf{v}_{\rm cm}$$

- **Acceleration of CM** (m/s^2): $$\mathbf{a}_{\rm cm} = \frac{d\mathbf{v}_{\rm cm}}{dt} = \ddot{\mathbf{r}}_{\rm cm}$$

- **Force on CM** (N=kg·m/s^2): $$\sum \mathbf{F}_{ext} = M\mathbf{a}_{\rm cm} = \frac{d\mathbf{p}_{\rm cm}}{dt}$$

All the laws and notions pertaining to the point particle described earlier applies to the total mass (M) as if applied to the center of mass (CM) of the extended body. This includes impulse, work, kinetic energy, power, potential energy, etc. of the extended body as it pertains to the translational motion of the center of mass.

- **Impulse on CM** (N·s): $$\mathbf{J}_{\rm cm} = \int_{t_i}^{t_f} \mathbf{F}dt' = \Delta \mathbf{p}_{\rm cm}$$

- **Work on CM** (J=N·m): $$W_{\rm cm} = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F}\cdot d\mathbf{r}_{\rm cm}$$

- As the notion of potential energy comes from work performed by an external conservative force, **there can be potential energy, as before, associated with M at the position of the center of mass**. For example, the gravitational potential energy is Mgh_{cm}.

- **Power on CM** (W=J/s):

$$P_{\mathrm{cm}} = \mathbf{F}_{ext} \cdot \mathbf{v}_{\mathrm{cm}}; \text{ generally, } P_{tot} = \frac{dE_{tot}}{dt}$$

- **Kinetic Energy of CM** (J): $K_{\mathrm{cm}} = \frac{1}{2} M v_{\mathrm{cm}}^2$

Rotations

Newton's laws still apply to every mass element of the extended body, so there is motion, momentum, work, energy (potential and kinetic), etc. associated to the motions about the center of mass of the body. For a rigid body, that motion takes the form of rotations. Relaxing the rigidity assumption, objects may deform, there may be random motion of molecules that comprise the body (heat and temperature), and there may be motions of the individual constituents of the molecules with respect to each other (vibration). There is work and energy associated with each of these motions. When dealing with liquids, gases, or plasmas (a gas comprised of charges particles) each element obeys Newton's laws and has an energy and momentum. Such systems will be described by a distribution function describing the position and velocities of the constituents. By integration, properties of the distribution, such as pressure (force per unit area), bulk velocity (akin to the velocity of the center of mass), etc. can be computed.

At this point we will consider rigid bodies and rotation. (The notions, if properly applied pertain generally, except for those involving the moment of inertia which applies to the rigid body.)

- **Angular Position** (radian): θ

- **Angular Displacement** (radian): $\Delta\theta = \theta_f - \theta_i$

Note: θ does not behave as a vector. However, the differential angular displacement $\overrightarrow{d\theta}$ does behave as a vector. Its direction obeys the "right-hand rule", where you curl your fingers in the direction that the angle changes, and your thump points in the direction of the vector. For counterclockwise rotations in a horizontal plane, the thumb points up; for clockwise rotations, the thumb points down.

- **Differential Angular Displacement** (radian):

$$\overrightarrow{d\theta}$$

- **Angular Velocity** (rad/s = s^{-1}): $$\overrightarrow{\omega} = \frac{d\overrightarrow{\theta}}{dt}$$

- **Angular Acceleration** (rad/s^2=s^{-2}):

$$\overrightarrow{\alpha} = \frac{d\overrightarrow{\omega}}{dt} = \frac{d(d\overrightarrow{\theta})}{dt^2}$$

- If α = constant, then we have $\ddot{\theta} = \alpha = constant$ in analogy to $\ddot{\mathbf{r}} = \mathbf{a} = constant$ for linear motion. The solution is

$$\omega = \omega_0 + \alpha t; \;\; \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

Note that time can be eliminated from these equations to yield:

$$\omega_f^2 = \omega_\circ^2 + 2\alpha\theta$$

- **Arc Length** (m): $$s = r\theta$$

Angular displacement is measured in **radians**. *Do not use degrees.* When $s = r$, $\theta = 1$ radian. Thus, 2π radians is the same as 360°. For rigid body rotation, $d\overrightarrow{\theta}$, $\overrightarrow{\omega}$, and $\overrightarrow{\alpha}$ all point along the axis of rotation following the "right-hand rule".

- **Differential Arc Length** (m): $$d\mathbf{s} = d\overrightarrow{\theta} \times \mathbf{r}$$

- **Tangential Velocity** (m/s): $$\mathbf{v}_t = \frac{d\mathbf{s}}{dt} = \overrightarrow{\omega} \times \mathbf{r}$$

- **Tangential Acceleration** (m/s^2): $$\mathbf{a}_t = \overrightarrow{\alpha} \times \mathbf{r}$$

There may also be a radial velocity and acceleration. *The total linear velocity (acceleration) is the vector sum of the tangential velocity (acceleration) and the radial velocity (acceleration).* If $\mathbf{r}$ represents the radius vector of the circle on which the particle (or mass element) is at that instant moving, then the radial acceleration is just the centripetal acceleration, and $a = \sqrt{a_t^2 + a_c^2} = r\sqrt{\alpha^2 + \omega^4}$.

- **Moment of Inertia** (kg·m^2):

$$I = \sum_i m_i r_i^2 = \int r^2 dm = \int \rho r^2 dV$$

- **Parallel-Axis Theorem:** Suppose the moment of inertia about an axis through the center of mass is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

$$I = I_{CM} + MD^2$$

- **Angular Momentum** (kg·m²/s): $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

For a rigid body rotating about an axis

$$\mathbf{L} = I\overrightarrow{\omega}$$

If a force is applied to a rigid object pivoted about an axis, then the object tends to rotate about that axis. The tendency of a force to rotate an object is the **torque**.

- **Torque** (N·m): $\overrightarrow{\tau} = \mathbf{r} \times \mathbf{F} = \dfrac{d\mathbf{L}}{dt} = rF\sin\phi; \;\; r\sin\phi = moment\;\; arm$

Torque should not be confused with force. Forces can cause a change in linear motion, as described by Newton's second law. Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces, in the combination that is called *torque*. *Do not confuse torque with work, which have the same units but are very different concepts.* However, it is the torque that changes the angular momentum and results in angular acceleration of a rigid body

$$\sum \overrightarrow{\tau}_{ext} = I\overrightarrow{\alpha} = \frac{d\mathbf{L}_{tot}}{dt}$$

- **Rotational Work** (J):

$$dW_R = \mathbf{F} \cdot \overrightarrow{ds} = \mathbf{F} \cdot (\overrightarrow{d\theta} \times \mathbf{r}) = (\mathbf{r} \times \mathbf{F}) \cdot \overrightarrow{d\theta} = \overrightarrow{\tau} \cdot \overrightarrow{d\theta}$$

- **Rotational Power** (J/s): $P_R = \dfrac{dW_R}{dt} = \overrightarrow{\tau} \cdot \overrightarrow{\omega}$

Now, $dW_R = \sum \overrightarrow{\tau} \cdot \overrightarrow{d\theta} = I\overrightarrow{\alpha} \cdot \overrightarrow{d\theta} = I\alpha d\theta = I\frac{d\omega}{d\theta}\frac{d\theta}{dt}d\theta = I\omega d\omega$. The total work done by the net external force causing a system to rotate is

$$\sum W_R = \int_{\omega_i}^{\omega_f} I\omega d\omega = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

- **Kinetic Energy of Rotation** (J): $K_R = \frac{1}{2} I \omega^2$

The **total kinetic energy of a rigid body** is the sum of the kinetic energies of all the particles. This is the kinetic energy of the translational motion of the total mass M located at its center of mass plus the kinetic energy of rotation about the center of mass

$$K = \frac{1}{2} I_{\rm cm} \omega^2 + \frac{1}{2} M v_{\rm cm}^2$$

- **Condition for Rolling Motion:** $v_{\rm cm} = \omega R$

- **Conservation of Angular Momentum** states that the total angular momentum of an isolated system (i.e., a system subject to no external torque) at all times equals its initial momentum.

Thus, so far we have that, **for an isolated system, the total energy, momentum, and angular momentum each remain constant**.

- The two necessary conditions for an object to be in **equilibrium** is that (1) the net external force must equal zero

$$\sum \mathbf{F} = 0 \quad \text{(translational equilibrium)}$$

and (2) that the net external torque about *any* axis be zero

$$\sum \vec{\tau} = 0 \quad \text{(rotational equilibrium)}$$

Note that if an object is in translational equilibrium and the net torque is zero about one axis, then the net torque must be zero about any other axis.

- **Hooke's Law and Simple Harmonic Motion**. Any physical system which undergoes a small displacement from a stable equilibrium position is subject to a restoring force proportional to the displacement from its equilibrium position known as **Hooke's law**:

$$\mathbf{F} = -k\mathbf{x}$$

In one dimension, $a = \ddot{x} = -\omega^2 x$. The solution is the condition for **simple harmonic motion**; the system will oscillate about its equilibrium position according to

$$x = A\cos(\omega t - \phi)$$

where A is the amplitude of the displacement, $\omega = 2\pi f = 2\pi/T$ is the angular frequency of the oscillations, f and $T = 1/f$ are the frequency and period of the oscillation, and ϕ is a possible phase shift of the wave. Example systems obeying Hooke's law include the spring ($\omega = \sqrt{k/m}$) and the simple pendulum ($\omega = \sqrt{g/L}$).

- **Wave Speed:** $$v = f\lambda$$

where f is the frequency and λ is the wavelength of the wave.

- **Speed of Sound:** $$v = \sqrt{\frac{\gamma p}{\rho}}, \text{ or } v = \sqrt{\frac{B}{\rho}}$$

where γ is the ratio of specific heats (adiabatic index), p is the pressure, B is the bulk modulus, and ρ is the mass density.

- **Waves on a String:** $$v = \sqrt{\frac{T}{\mu}}$$

where T is the tension on the string, and μ is the mass per unit length of the string.

- **Standing Waves:**

$$f_n = \frac{nv}{2L}, \text{ where } n = 1, 2, 3, ... \text{ (fixed ends)}$$

$$f_n = \frac{(2n-1)v}{4L}, \text{ where } n = 1, 2, 3, ... \text{ (one open end)}$$

The lowest frequency, corresponding $n = 1$, is called the *fundamental frequency*. The frequency corresponding to $n = 2$ is the *first harmonic*, $n = 3$ is the *second harmonic*, etc...