Problem 1: Consider the one-dimensional (1D) harmonic oscillators described by the Hamiltonian:

\[ \hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2x^2, \]

where \( m \) is the mass of the particle, \( \omega \) is the angular frequency and \( \hat{p}_x \) is the linear momentum operator in the \( x \) direction. The allowed energy eigenvalues are: \( E_n = \hbar\omega(n + 1/2) \) where \( n = 0, 1, 2, \ldots \). The normalized eigenfunctions are:

\[ \Phi_n(x) = N_n \exp\left(-\frac{\alpha^2x^2}{2}\right) H_n(\alpha x) ; \quad N_n = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^n n!}}, \]

where \( N_n \) is the normalization constant, \( \alpha = \sqrt{m\omega/\hbar} \) is a parameter with the dimensionality of an inverse length and \( H_n(\alpha x) \) are Hermite polynomials. Calculate \( (\Delta x)^2 = \langle x^2 \rangle - (\langle x \rangle)^2 \) and \( (\Delta p_x)^2 = \langle \hat{p}_x^2 \rangle - (\langle \hat{p}_x \rangle)^2 \) for an arbitrary quantum state \( \Phi_n(x) \). Verify whether the Heisenberg uncertainty principle, \( (\Delta x)^2 (\Delta p_x)^2 \geq (\hbar/2)^2 \) is satisfied. \textbf{Hint:} Recall that \( \langle x \rangle = 0 \) and \( \langle \hat{p}_x \rangle = 0 \), so you need to calculate only \( \langle x^2 \rangle \) and \( \langle \hat{p}_x^2 \rangle \). The final result should be: \( (\Delta x)^2 (\Delta p_x)^2 = \hbar^2(n + 1/2)^2 \).
**Problem 2**: Consider the displaced one-dimensional (1D) harmonic oscillator:

\[ \hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega^2(x - x_0)^2, \]

where \( m \) is the mass of the particle, \( \omega \) is the angular frequency, \( \hat{p}_x \) is the linear momentum operator in the \( x \) direction and \( x_0 \) is the coordinate of the center of the 1D oscillator. Prove that the allowed energy eigenvalues are: \( E_n = \hbar \omega (n + 1/2) \) where \( n = 0, 1, 2, \ldots \) and the normalized eigenfunctions are:

\[ \Phi_n(x - x_0) = N_n \exp \left( -\frac{\alpha^2(x - x_0)^2}{2} \right) H_n[\alpha(x - x_0)] ; \quad N_n = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^n n!}}, \]

where \( N_n \) is a normalization constant that has been previously defined, \( \alpha = \sqrt{m\omega/\hbar} \) is a parameter with the dimensionality of an inverse length and \( H_n(z) \) are the Hermite polynomials with \( z \) as argument.
**Problem 3:** Consider a two-dimensional (2D) isotropic harmonic oscillator described by the Hamiltonian:

\[
\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2) ,
\]

where \(m\) is the mass of the particle, \(\omega\) is the angular frequency, and \(\hat{p}_x, \hat{p}_y\) are the respective linear momentum operators in the \(x\) and \(y\) direction. Prove that the allowed energy eigenvalues are of the form: 

\[
E_{n_x n_y} = \hbar\omega (n_x + n_y + 1) \text{ where } n_x = 0, 1, 2, \ldots \text{ and } n_y = 0, 1, 2, \ldots
\]

Note \(n = n_x + n_y = 0, 1, 2, \ldots\) Find the degeneracy of any given energy eigenvalue, \(E_{n_x n_y n_z}\) in terms of quantum number, \(n\). Verify that the degeneracy of any given energy eigenvalue \(E_{n_x n_y}\) in terms of number \(n\) is: 

\[
D_n = (n + 1).
\]

**Note:** If degeneracy is one, that means that the energy eigenvalue is **non degenerate**.
Problem 4: Consider a three-dimensional (3D) isotropic harmonic oscillator described by the Hamiltonian:

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2),$$

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where $m$ is the mass of the particle, $\omega$ is the angular frequency, and $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ are the respective linear momentum operators in the $x$, $y$ and $z$ direction. Prove that the allowed energy eigenvalues are of the form:

$$E_{n_x,n_y,n_z} = \hbar\omega(n_x + n_y + n_z + 3/2)$$

where $n_x = 0, 1, 2, \ldots$, $n_y = 0, 1, 2, \ldots$ and $n_z = 0, 1, 2, \ldots$. Note $n = n_x + n_y + n_z = 0, 1, 2, \ldots$ Find the degeneracy of any given energy eigenvalue, $E_{n_x,n_y,n_z}$ in terms of quantum number, $n$. Note: If degeneracy is one, that means that the energy eigenvalue is non degenerate.
**Problem 5:** Consider two identical one-dimensional (1D) harmonic oscillators. The Hamiltonian of the two particles in oscillatory motion is:

$$\hat{H} = \frac{\hat{p}_{x_1}^2}{2m} + \frac{1}{2} m \omega^2 x_1^2 + \frac{\hat{p}_{x_2}^2}{2m} + \frac{1}{2} m \omega^2 x_2^2,$$

where the indexes 1 and 2 refer respectively to particle 1 and 2. The energy eigenvalues are: $E_{n_1n_2} = \hbar \omega (n_1 + n_2 + 1)$ where $n_1 = 0, 1, \ldots$ and $n_2 = 0, 1, \ldots$ The eigenfunctions corresponding to those eigenvalues are: $\Psi_{n_1n_2}(x_1, x_2) = \Phi_{n_1}(x_1) \Phi_{n_2}(x_2)$ where $\Phi_{n_i}(x_i)$ are the normalized eigenfunctions for the 1D oscillator for particle $i = 1, 2$ respectively. The groundstate wave function is: $\Psi_{00}(x_1, x_2)$ and corresponds to the lowest energy $E_{00} = \hbar \omega$. Find the “average relative distance” between particle 1 and 2 in the groundstate:

$$\langle |x_1 - x_2| \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 |x_1 - x_2| \Psi_{00}(x_1, x_2)^\ast \Psi_{00}(x_1, x_2)$$