Evard and Jafari (1992) established the following theorem:

Theorem 2.1. (Evard & Jafari). Let *f* be a holomorphic function defined on an open convex subset D_f of $\mathbb{C}$. Let $a,b\in D_f$ be such that $a\neq b$ and $f(a)=f(b)=0$. Then there exists $z_1,z_2\in\left]a,b\right[$ such that $\Re\left(f'(z_1)\right)=0$ and $\Im\left(f'(z_2)\right)=0$.

So, their result requires complete analyticity of the function, a convex domain to ensure a total containment of the convex hull of every pair of points of interest, and then depart quite a bit from the classical format of Rolle's conclusion with two points where only single components of the original function satisfy local tangential flatness.

In the sequel, we return to the more conventional setting with less strenuous requirements on the function and pick a single point of directional tangency for the whole complex function with the added twist of arbitrary path connection between isopotent points.

Theorem 2. 2. Let f represent a $\mathbb{C}$-valued differentiable function defined on an open path-connected subset Ω of $\mathbb{C}$. Let $a,b\in\Omega$ two distinct points such that $f(a)=f(b)$. Then, for any smooth simple connected path $\Gamma\subset\Omega$ from *a* to *b* with canonical parametrization γ, there is some $z_0\in\Gamma$ such that $f'(z_0)\gamma'\left(\gamma^{-1}(z_0)\right)\bullet(b-a)=0$.

Proof:
Let $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ be a differentiable function and $\gamma:[0,1]\to\Gamma$ a smooth parametrization of $\Gamma\subset\Omega$, with $\gamma(0)=a$ and $\gamma(1)=b$. Now, define a composite projection $\Delta:[0,1]\to\mathbb{R}$ by $\Delta(t)=f(\gamma(t))\bullet(b-a)$, where $\bullet$ denotes the scalar product in the Hilbert space structure of $\mathbb{C}$. Clearly, Δ satisfies all the hypotheses of the classical Rolle's theorem; Indeed, Δ is obviously smooth on $(0,1)$, continuous on $[0,1]$, by composition and $\Delta(0)=f(\gamma(0))=f(a)=f(b)=f(\gamma(1))=\Delta(1)$. Hence, choose $t_0\in(0,1)$ and let $z_0=\gamma(t_0)\in\Gamma$ such that $f'(z_0)\gamma'(t_0)\bullet(b-a)=0$.

Remark 2. 3.
As a geometric interpretation, we can note that the image of this particular type of map is a 2D-complex manifold. Theorem2.2 says that when two points $a\neq b$ on that surface have the same height, then along any path (entirely contained in the surface) connecting the two points, we can find at least one point where the tangency is flat, at least in the direction of b - a.

Theorem 2.4. Let f represent a $\mathbb{C}$-valued differentiable function defined on an open path-connected subset Ω of $\mathbb{C}$. Let $a,b\in\Omega$ be two distinct points. Then, for any smooth simple connected path $\Gamma\subset\Omega$ from a to b with canonical parametrization γ, there exists some $z_0\in\Gamma$ such that $\left[f'(z_0)-\frac{f(b)-f(a)}{b-a}\right]\gamma'\left(\gamma^{-1}(z_0)\right)\bullet(b-a)=0$.

Proof:
Given $f:\Omega\subset\mathbb{C}\to\mathbb{C}$ satisfying the above hypotheses, we introduce

$$\Phi_f(z)=f(z)-f(a)-\frac{f(b)-f(a)}{b-a}(z-a),$$

whose (2×2) Jacobian is given by

$$D\Phi_f(z)=Df(z)-\frac{f(b)-f(a)}{b-a}$$

$$\equiv\begin{bmatrix}\frac{\partial f_1}{\partial x}(z)-\Re\left\{\frac{f(b)-f(a)}{b-a}\right\} & \frac{\partial f_2}{\partial x}(z)-\Im\left\{\frac{f(b)-f(a)}{b-a}\right\}\\ \frac{\partial f_1}{\partial y}(z)-\Re\left\{\frac{f(b)-f(a)}{b-a}\right\} & \frac{\partial f_2}{\partial y}(z)-\Im\left\{\frac{f(b)-f(a)}{b-a}\right\}\end{bmatrix},$$

where f_1 and f_2 respectively denote the real and imaginary components of f, $z = x + i\,y$, and such that $\Phi_f(a)=\Phi_f(b)$ = 0. And thus, by theorem 2.2 above, we get

$$\left(D\Phi_f(z_0)\right)\gamma'\left(\gamma^{-1}(z_0)\right)\bullet(b-a)=0.$$

Remark 2. 5.
Similar to remark 2.3 above, a corresponding geometrical interpretation is made for the existence of some tangent hyperplane parallel to the support of any fixed direction through the convex hall of the manifold.

3. Numerical Illustration

The holomorphic function $f(z)=e^{2\pi iz}-1$ has zeroes at every integral value $z=0,\pm1,\pm2,\ldots$, yet its Jacobian $f'(z)=-2\pi e^{-2\pi y}\begin{bmatrix}\sin 2\pi x & -\cos 2\pi x\\ \cos 2\pi x & \sin 2\pi x\end{bmatrix}$ never vanishes anywhere on the complex plane and thus cannot satisfy the conclusions of the classical Rolle's theorem.

On the other hand,
$\alpha)$ if $\Gamma=[-1,1]\times\{0\}$, path connecting the pair of points a = (- 1,0) and b = (1,0), then we have b - a = (2,0) as well as the trivial parametrization $\gamma(t)=(2t-1,0)$ with $\gamma'(t)=(2,0)\equiv\begin{bmatrix}2\\0\end{bmatrix}$ in matrix form. On Γ, we can easily solve that $\left[f'(x_0,0)\gamma'(t_0)\right]\bullet\begin{bmatrix}2\\0\end{bmatrix}=0$ at any $z_0=(n,0)\in\mathbb{Z}\times\{0\}$; so that the origin can be selected as solution;

$\beta)$ $\Gamma = semicircle\ of\ radius 1, at\ the\ origin$, as path connecting the same 2 points *a* = (-1,0) and *b* = (1,0). This time, a natural parametrization could be $\gamma(t) = (\cos\pi(1-t), \sin\pi(1-t))$ with $\gamma'(t) = \pi(\sin\pi(1-t), -\cos\pi(1-t))$. Then, it is easy to solve that $z_0 = \gamma(1/2) = (0,1)$, and then verify that

$$\left[f'(z_0)\gamma'(1/2) \right] \bullet \begin{bmatrix} 2 \\ 0 \end{bmatrix} = -2\pi^2 e^{-2\pi} \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \bullet \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0; \text{ and}$$

$\gamma)$ More generally, for any smooth simple connected path $\Gamma \subset \Omega$ from (-1, 0) to (1, 0) with smooth parametrization γ, let $\gamma(0) = (-1,0)$, $\gamma(1) = (1,0)$, and $\gamma(t) = (x, y) = z$; then, $f(\gamma(t)) = (e^{-2\pi y}\cos 2\pi x - 1, e^{-2\pi y}\sin 2\pi x)$. We consider the special smooth real-valued map $\Delta(t) = 2e^{-2\pi y}\cos 2\pi x$, due to the smoothness of $\gamma(t) = (x, y)$. Clearly, $\Delta(0) = \Delta(1) = 2$; which is enough to choose some t_0 such that

$$\Delta'(t_0) = f'(\gamma(t_0))\gamma'(t_0) \bullet \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 0.$$

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