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ABSTRACT

We introduce a class of Calderon-Zygmund Lp-integral operators and study some conditions for boundedness on Lebesgue reflexive spaces.

1. Introduction

An important classical problem in potential theory has been that of the study of all possible analytic properties of the integral operators occurring in representation problems via transforms, with direct implications in quantum mechanics or particle physics. In the fifties, Calderon and Zygmund studied classes of singular integral operators ([2], [3],…) that involved almost all the classical kernels of the Newtonian and logarithmic type potentials met in Hilbert transform, Poisson, and many others. They were mostly concerned with logarithmically bounded kernels $\left|\Omega(x,y)\right| \le \frac{C}{|x-y|}$, whose differentials was boundedly Newtonian $\left|\frac{\partial\Omega(x,y)}{\partial x}\right| + \left|\frac{\partial\Omega(x,y)}{\partial y}\right| \le \frac{C}{|x-y|^2}$. In the sequel, we

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consider potentials such as $\Omega(x,y)=\dfrac{1}{(x-y)^{1+\lambda}}$ ($\lambda \geq 0$), $\Omega(x,y)=\exp\left(\dfrac{1}{x-y}\right)$ on the half-space $x \leq y$, $\Omega(x,y)=\sin(x-y)^{-\lambda}+\cos(x-y)^{-\lambda}$ $(\lambda \geq 1)$,…where the boundedness condition is extended to the first fixed-N derivatives. Then, N=0 will recover the Hilbert transform, while N=1 represents the Calderon-Zygmund class.

2. Boundedness Conditions

A very useful one-dimensional formulation of F. Riesz decomposition theorem can be found in A. Zygmund treatise on trigonometric series [5], p.242.

Theorem2.1 (F.Riesz Decomposition):

Let $f \in L^p(\mathbb{R})$, $p \geq 1$ integer, and $\alpha > 0$. There exist a decomposition $\mathbb{R} = G \cup B$ and two sequences $\{G_j\}_j$ and $\{B_k\}_k$ of subintervals such that

(i) $f(x) \leq \alpha$ a.e.

(ii) $G=\bigcup_j G_j$; $\forall x \in G$, *there is a nested subsequence* $G_{j_1} \supset G_{j_2} \supset \ldots$ *shrinking to* $\{x\}=\bigcap G_{j_\gamma}$.

(iii) $B=\bigcup_k B_k$ *non-overlapping intervals satisfying* $\alpha \leq \dfrac{1}{|B_k|}\int_{B_k} f(x)\,dx \leq 2\alpha$, *for all k.*

Let $B(x,\delta)=\{y \in \mathbb{R}^n; |x-y|<\delta\}$ denote the open ball of radius δ in $\mathbb{R}^n$, in general.

Lemma2.2

Let $x, y \in \mathbb{R}$, $\delta > 0$. If $\Omega(x,y)$ is (N+1) times continuously differentiable with $\left|\dfrac{\partial^n \Omega(x,y)}{\partial x^n}\right| + \left|\dfrac{\partial^n \Omega(x,y)}{\partial y^n}\right| \leq \dfrac{K}{|x-y|^{n+1}}$, $\forall n = 0,1,2,\ldots,N$, for some constant K>0, then for any sequence $\{y_k\} \subset \mathbb{R} - B(x,\delta)$, there exists a constant C>0 such that $\int_{|x-y_k|\geq\delta} |\Omega(x,y)-\Omega(x,y_k)|\,dx \leq C$.

Proof:

Applying the Lagrange form of the *N*th Taylor remainder theorem, we get a first local estimate

$$\left|\Omega(x,y)-\Omega(x,y_k)\right|$$

$$\leq\left|\frac{\partial\Omega}{\partial y}(x,y_k)\right|\left|y-y_k\right|+\left|\frac{\partial^2\Omega}{\partial y^2}(x,y_k)\right|\frac{\left|y-y_k\right|^2}{2!}+\ldots+\left|\frac{\partial^N\Omega}{\partial y^N}(x,y_k)\right|\frac{\left|y-y_k\right|^N}{N!}$$

$$+\ \frac{\left|y-y_k\right|^N}{N!}\int_0^1\left|\frac{\partial^{N+1}\Omega}{\partial y^{N+1}}\left(x,y_k+\tau(y-y_k)\right)\right|(1-\tau)^N\,d\tau$$

$$\leq\frac{K}{\left|x-y_k\right|^2}\left|y-y_k\right|+\frac{K}{\left|x-y_k\right|^3}\frac{\left|y-y_k\right|^2}{2!}+\ldots+\frac{K}{\left|x-y_k\right|^{N+1}}\frac{\left|y-y_k\right|^N}{N!}+K\frac{\left|y-y_k\right|^N}{N!}\int_0^1\frac{(1-\tau)^N}{\left|(x-y_k)-\tau(y-y_k)\right|^{N+1}}d\tau$$

We introduce new variables $x-y_k=\eta r_k$ and $y-y_k=\xi r_k$ for each k in order to get

$$\int_{\left|x-y_k\right|\geq\delta}\left|\Omega(x,y)-\Omega(x,y_k)\right|dx$$

$$\leq K\left|y-y_k\right|\int_{\left|x-y_k\right|\geq\delta}\frac{dx}{\left|x-y_k\right|^2}+K\frac{\left|y-y_k\right|^2}{2!}\int_{\left|x-y_k\right|\geq\delta}\frac{dx}{\left|x-y_k\right|^3}+\ldots+K\frac{\left|y-y_k\right|^N}{N!}\int_{\left|x-y_k\right|\geq\delta}\frac{dx}{\left|x-y_k\right|^{N+1}}+$$

$$+\ K\frac{\left|y-y_k\right|^N}{N!}\int_{\left|x-y_k\right|\geq\delta}\int_0^1\frac{(1-\tau)^N}{\left|(x-y_k)-\tau(y-y_k)\right|^{N+1}}d\tau dx$$

$$\leq K\left[\left|\xi\right|\int_{\left|\eta\right|\geq\delta}\frac{d\eta}{\left|\eta\right|^2}+\frac{\left|\xi\right|^2}{2!}\int_{\left|\eta\right|\geq\delta}\frac{d\eta}{\left|\eta\right|^3}+\ldots+\frac{\left|\xi\right|^N}{N!}\int_{\left|\eta\right|\geq\delta}\frac{d\eta}{\left|\eta\right|^{N+1}}+\frac{\left|\xi\right|^N}{N!}\int_{\left|\eta\right|\geq\delta}\int_0^1\frac{d\tau}{\left|\eta-\tau\xi\right|^{N+1}}d\eta\right]\leq C<\infty$$

since every integral avoids the only singularity at the origin.

The proof of our main theorem requires the use of a couple of classical results.

<u>Definition2.3</u>: A *Calderon-Zygmund operator* $Tf(x)=\int\Omega(x,y)f(y)dy$ is defined on L^2 by a kernel that satisfies two boundedness conditions (*i*) $\left|\Omega(x,y)\right|\leq\dfrac{C}{\left|x-y\right|}$ and (*ii*) $\left|\frac{\partial\Omega(x,y)}{\partial x}\right|+\left|\frac{\partial\Omega(x,y)}{\partial y}\right|\le\frac{C}{|x-y|^2}$.

Theorem2.4 (Calderon-Zygmund, [3])

Let T be a given Calderon-Zygmund operator and a Riesz decomposition $\mathbb{R}^N=G\cup B$. *Let* $f\in L^1$ and $\alpha>0$. *Then, there exists a constant C>0 such that*

$\left|\left\{x;\left|T\left(f/G\right)(x)\right|\ge\alpha/2\right\}\right|\le C\alpha^{-1}\|f\|_{L^1}$.

Let $1<q_1\le p_1$ and $1<q_2\le p_2$, integers; 0<*t*<1, $p=\left(\frac{t}{p_1}+\frac{1-t}{p_2}\right)^{-1}$ and $q=\left(\frac{t}{q_1}+\frac{1-t}{q_2}\right)^{-1}$.

Let $\|f\|_{L^q_{\text{weak}}}\equiv\operatorname*{Inf}_{\alpha>0}\left\{\alpha\left|\left\{x;\left|f(x)\right|\ge\alpha\right\}\right|^{1/q}\right\}$.

Theorem2.5 (Marcinkiewicz Interpolation)

Let T be an operator such that $\|Tf\|_{L^{q_1}_{\text{weak}}}\le A\|f\|_{L^{p_1}}$ *and* $\|Tf\|_{L^{q_2}_{\text{weak}}}\le B\|f\|_{L^{p_2}}$ *with A and B positive constants. Then,* $\|Tf\|_{L^q}\le C\|f\|_{L^p}$ *for some constant C>0.*

In the following theorem, for sake of simplicity when $n=0$, we mean

$$\left|\frac{\partial^n\Omega(x,y)}{\partial x^n}\right|+\left|\frac{\partial^n\Omega(x,y)}{\partial y^n}\right|\equiv\left|\Omega(x,y)\right|.$$

Theorem2.6

Let $N\ge1$. *Let T denote a linear integral operator with kernel* $\Omega(x,y)$ *such that*

$\left|\frac{\partial^n\Omega(x,y)}{\partial x^n}\right|+\left|\frac{\partial^n\Omega(x,y)}{\partial y^n}\right|\le\frac{K}{|x-y|^{n+1}}$, $\forall n=0,1,2,\ldots,N$ *, for some constant K>0 .*

Then, T is a bounded endomorphism on any reflexive $L^p(\mathbb{R})$, $p>1$.

Proof:

We define

$$g(x)=\begin{cases} f(x), & x\in G\\ \frac{1}{|B_k|}\int_{B_k}f(u)\,du, & x\in B_k\end{cases}$$ and

$$h(x)=\begin{cases}0, & x\in G\\ f(x)-\frac{1}{|B_k|}\int_{B_k} f(u)\,du, & x\in B_k\end{cases}$$

It follows that $f(x)=g(x)+h(x)$ a.e. By Calderon-Zygmund's theorem above, we get

an upper bound estimate $\left|\{x:|Tg(x)|\geq \alpha/2\}\right|\leq C_1\alpha^{-1}\|f\|_{L^1}$.

Next, we set $B_k=[c_k-r_k,c_k+r_k]$ and $B_k^*=[c_k-2r_k,c_k+2r_k]$, $B^*=\bigcup_k B_k^*$, and also

$G^*=\mathbb{R}-B^*$. Then, $\|f\|_{L^1}\geq\int_{B_k}|f(x)|dx\geq\alpha|B_k|$ implies that

$\left|\{x\in B^*:|Th(x)|\geq\alpha/2\}\right|\leq |B^*|\leq\frac{2}{\alpha}\|f\|_{L^1}$. On the other hand, we have

$$\int_{G^*}|Th(x)|dx\geq\int_{\{x\in G^*;|Th(x)|\geq\alpha/2\}}|Th(x)|dx\geq\frac{2}{\alpha}\left|\{x\in G^*;|Th(x)|\geq\alpha/2\}\right|.$$

In order to find an upper bound for $\int_{G^*}|Th(x)|dx$, we define a new sequence

$$h_k(x)=\begin{cases}0, & x\notin B_k\\ f(x)-\frac{1}{|B_k|}\int_{B_k} f(u)\,du, & x\in B_k\end{cases}$$

of disjointly supported components of $h(x)=\sum_k h_k(x)$ a.e. But then,

$$\int_{G^*}|Th(x)|dx\leq\sum_k\int_{\mathbb{R}-B_k^*}Th_k(x)dx=\sum_k\int_{\mathbb{R}-B_k^*}\left|\int_{B_k}\Omega(x,y)h_k(y)dy\right|dx$$

$$=\sum_k\int_{\mathbb{R}-B_k^*}\left|\int_{B_k}[\Omega(x,y)-\Omega(x,c_k)]h_k(y)dy\right|dx$$

$$\leq\sum_k\int_{\mathbb{R}-B_k^*}\int_{B_k}|\Omega(x,y)-\Omega(x,c_k)|h_k(y)dydx$$

since

$$\int_{B_k} h_k(y)\,dy = \int_{B_k}\left[f(y) - \frac{1}{|B_k|}\int_{B_k} f(z)\,dz \right] dy = \int_{B_k} f(y)\,dy - \frac{1}{|B_k|}\int_{B_k}\left[\int_{B_k} f(z)\,dz\right] dy = \int_{B_k} f(y)\,dy - \frac{1}{|B_k|}\int_{B_k} f(z)\,dz\,|B_k| = 0$$

We apply Lemma to get $\int_{\mathbb{R}-B_k^*}\left|\Omega(x,y)-\Omega(x,c_k)\right|dx \le C_2 < \infty$. It follows that

$$\int_{G^*}\left|Th(x)\right|dx \ \le C_2 \sum_k \int_{B_k}\left|h_k(y)\right|dy$$

$$\le C_2 \sum_k \int_{B_k}\left[\left|f(y)\right| + \frac{1}{|B_k|}\int_{B_k}\left|f(z)\right|dz\right]dy \ \le 2C_2 \sum_k \int_{B_k}\left[\left|f(y)\right|\right]dy \le 2C_2 \,\|f\|_{L^1}.$$

Hence, $\left|\left\{x \in G^*; \left|Th(x)\right| \ge \alpha/2\right\}\right| \ \le \frac{4}{\alpha} C_2 \,\|f\|_{L^1}$ and in particular,

$$\left|\left\{x; \left|Th(x)\right| \ge \alpha/2\right\}\right| \le \frac{2}{\alpha}\|f\|_{L^1} + \frac{4}{\alpha}C_2\|f\|_{L^1} \equiv \frac{C_3}{\alpha}\|f\|_{L^1}, \text{ and}$$

$$\left|\left\{x; \left|Tf(x)\right| \ge \alpha\right\}\right| = \left|\left\{x; \left|Tg(x)\right| \ge \alpha/2\right\}\right| + \left|\left\{x; \left|Th(x)\right| \ge \alpha/2\right\}\right| \le \frac{C_1}{\alpha}\|f\|_{L^1} + \frac{C_3}{\alpha}\|f\|_{L^1} \equiv \frac{C}{\alpha}\|f\|_{L^1}$$

And thus we get $\|Tf\|_{L^1_{\text{weak}}} \ \le \ C\|f\|_{L^1}$.

T is finally extended as a bounded endomorphic operator on any $L^p(\mathbb{R})$ $1 < p < \infty$ according to Marcinkiewicz interpolation theorem, combined with a norm duality operation on the adjoint of T.

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