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with Delayed-Service**

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A Single-server Poisson Queueing System with Delayed-Service

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Abstract

To set the tone for more complicated models that will appear later, in this paper we investigate busy period and steady-state and transient queue length of a single-server Poisson queue with delayed-service. We will analyze this model by considering $M/G/1$ as well as the use of differential difference equations approximating a non-Markovian system. We obtain the distribution of the length of a busy period, steady-state mean and distribution of the queue length, Laplace transform of the probability of the system being empty and Laplace transform of the generating function of the distribution of the transient queue length.

Key Words: Queueing System, Single-server, Poisson, Non-Markovian, Delayed-Service

1. Introduction

Often in queueing literature, “delay” and “waiting time” are used interchangeably. In what follows, we will make a clear difference between these two characteristics of the queueing system under study. A limited number of papers have appeared in the literature on delayed-feedback and delayed-busy period queues. Hannibalsson and Disney (1977) studied an $M/M/1$ queue with finite capacity with delayed-feedback. They considered the delay in the feedback as another a finite-capacity multiple-server queue and found the steady-state probabilities for the two dimensional Markov chain obtained through matrix operation. Foley and Disney (1983) considered queues with delayed-feedback and focused on performance measures such as queue length, busy period, and departure process. They studied a multi-server queue in detail by

considering the delay station as a single-server station in tandem with the multi-server station, but put them in parallel as an upper and a lower station. Medhi and Templeton (1992) considered a queueing system such that the server leaves the service station when it becomes empty and returns when the queue builds up to a pre-assigned level. They derived the average queue length and the distribution of the busy period for a steady-state $M/G/1$ with vacation.

Queues with delayed-service have not received much of attention in the literature. An example of this type of delay is a system that needs a setup or switchover time for each task to be processed, i.e., a task has to either wait until the server is warmed-up or it has to go through several procedures before its service starts. For instance, consider a screen printing operation where a single worker must produce garments with multiple colors. A setup time is needed each time a new color is printed.

Katsikopoulos and Engelbrecht (2003) considered a Markov decision process (MDP) that involves three types of delays, namely, observation delay, action delay and cost delay. They derived results for constant as well as for random delay by reducing MDP with delay to an MDP without delay, which differ only in the size of the state space. They consider MDPs with infinite decision-making horizon and costs adjust by a discounting factor. They minimize the total expected cost induced over the horizon. Courtois (1980) considered two variants of the $M/G/1$ finite capacity with delay. His first model studies the case when every busy period of the server is followed by the execution of a task different than the servicing of ordinary customers. He calls the time spent by the server to perform this task a delay. He used an embedded Markov chain approach to obtain the Laplace Stieltjes transform of the steady-state distribution of the queue length. He also found explicit solutions of the first and second moments of the queueing time distribution. As a second variant of his model, he considered the following case: if a delay has been completed and no customer has arrived, then the server starts a new delay instead of entering an idle period.

To set the tone for more complicated models that will appear later, in this paper we investigate busy period and steady-state and transient queue length of a single-server Poisson queue with delayed-service. By delay we mean the time between entering the service station until the task's service starts. This is different from the waiting time in the waiting room and the time it takes for the service to be completed. The amount of mathematics involved in development of this model is so great that although quite some work has been done on $M/G/1$, we feel it is worth going through this model in detail.

2. The Model

We consider an $M/G/1$ queueing system with a mean arrival rate λ . The processing time, Y_n , of the n^{th} task, $n = 1, 2, \dots$, consists of two independent parts, the delayed time, τ , and the service time, X_n . Thus, $Y_n = X_n + \tau$. That is, a task has to go through two stages to complete its processing. We assume that $\{Y_n\}$ is a sequence of identically distributed, mutually independent (*iid*), positive random variables with distribution $K(t)$. We also assume that the distribution for

the sequence $\{X_n\}$ is $H(t)$, where $H(t)$ is exponential with parameter μ_x . Thus, the cumulative distribution functions (CDFs) of X , representing $\{X_n\}$ and Y , representing $\{Y_n\}$, respectively, are $H(t) = P(X \leq t) = 1 - e^{-\mu_x t}$, $t > \tau$, and $K(t) = P(Y \leq t) = P(X + \tau \leq t)$, respectively.

Note that since a service cannot start before the delay period is over, probability of completion of the processing of a task before τ is zero. Thus, for the processing time, we will have $Y \geq \tau$.

Definition 1: By completion of a service with delay within the interval $(t, t + \Delta t)$, $t \geq \tau$, it is meant that processing of a task starts at $t = 0$, goes through a delay period τ , the service starts after the delay, and ends at $t + \Delta t$, where τ is a non-negative fixed delay time.

We consider two types of distributions for τ , namely, the exponential with parameter μ_τ and the deterministic.

2.1. Exponential Delay

In the first case, Y is a sum of two *iid* exponential random variables with the Laplace transform of its distribution $\left(\frac{\mu_x}{\mu_x + s}\right)\left(\frac{\mu_\tau}{\mu_\tau + s}\right)$. If $\mu_x \neq \mu_\tau$, then the probability density function, *pdf*, of Y will be $c_1 \mu_x e^{-\mu_x t} + c_2 \mu_\tau e^{-\mu_\tau t}$, with $c_1 + c_2 = 1$ (see Cox and Miller (1984, p. 257)). If $\mu_x = \mu_\tau = \mu$, then completion of a processing will require a task to undergo two stages with mean $\mu_x + \mu_\tau = 2\mu$. The system, in this case, will be a $M/E_2/1$ with parameter $\frac{1}{2\mu}$ (see Medhi (2003, p. 178)). Thus, the Laplace transform of the generating function of the transient distribution and the steady-state generating function of the queue size (see (4.2.17) and (4.2.9) of Medhi (2003)) are $G^*(z, s) = \frac{z - \mu(1-z)P_0^*(s)}{z(s + \mu + \lambda) - \mu - \lambda z^2(1+z)/2}$ and $G(z) = \frac{2\mu(1-\rho)}{2\mu - \lambda z(1+z)}$, respectively, where $\rho = \frac{\lambda}{\mu}$.

In the steady-state case we can readily have (Gross and Harris (1998, pp. 133-134)) expected measures of effectiveness such as the mean waiting time in the line, W_q , mean waiting time in the system, W , mean waiting queue length, L_q , and mean queue length, L , as:

$$W_q = \frac{3\rho}{2\mu(1-\rho)}, \quad W = \frac{L}{\lambda} = W_q + \frac{2}{\mu},$$

$$L_q = \lambda W_q = \frac{3\rho^2}{2\mu(1-\rho)}, \quad \text{and} \quad L = L_q + \rho.$$

2.2. Deterministic Delay

The rest of the papers will deal with the second case, i.e., deterministic delay time τ and exponential, with parameter μ , service time X . Thus, for the distribution of the processing time we have

$$\begin{cases} K(t) = P(Y \leq t) = P(X + \tau \leq t) = P(X \leq t - \tau) = 1 - e^{-\mu(t-\tau)}, & t \in [\tau, \infty) \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

That is, $K(t)$ is the shifted exponential distribution (see Hogg and Tanis (1993, p. 215)) with parameters (μ, τ) . Therefore, the mean processing time and the variance of the processing time are

$$E(Y) = \frac{1}{\mu} + \tau \text{ and } \sigma_Y^2 = \frac{1}{\mu^2}. \quad (2)$$

It is easy to see that the model described by (1) is not Markovian, because

$$P(Y > t+h | Y > t) = \frac{P(Y > t+h)}{P(Y > t)} = \frac{e^{-\mu(t+h-\tau)}}{e^{-\mu(t-\tau)}} = e^{-\mu h} \neq e^{-\mu(t+h-\tau)} = P(Y > h).$$

Our delayed-service single-server queueing system can be described as $M/G/1$, where G is the processing distribution $K(t)$ defined by (1). This will allow us to use properties of $M/G/1$ when appropriate.

Note also, that the cumulative distribution function, *CDF*, of the interdeparture process, $C(t)$, (see Gross and Harris (1998, 319)) is:

$$\begin{aligned} C(t) &= \lambda \left(\frac{1}{\mu} + \tau \right) (1 - e^{-\mu(t-\tau)}) + \left[1 - \lambda \left(\frac{1}{\mu} + \tau \right) \right] \int_{\tau}^t (1 - e^{-\mu(t-x-\tau)}) \lambda e^{-\lambda x} dx \\ &= \left[1 - \lambda \left(\frac{1}{\mu} + \tau \right) \right] (e^{-\lambda \tau} - e^{-\lambda t}) - \frac{\lambda e^{\mu \tau}}{\mu - \lambda} (e^{-\lambda \tau} - e^{-(\mu + \lambda)t}) \\ &\quad + \lambda \left(\frac{1}{\mu} + \tau \right) \left[1 - e^{-\mu(t-\tau)} + \frac{\lambda e^{\mu \tau}}{\mu - \lambda} (e^{-\lambda \tau} - e^{-\mu(t-\tau) - \lambda t}) \right]. \end{aligned}$$

Further, note that the utilization factor or traffic intensity, say ρ_X , for a stochastic process is defined as the probability that the system experiences no idleness during interdeparture period, i.e., $\rho = \frac{\text{mean service time of a task}}{\text{mean interarrival time}}$ (see Karlin and Taylor (1981, p. 499)). Thus, in our case,

$$\text{we have } \rho_X = \frac{\lambda}{\mu} \text{ and } \rho_Y = \lambda \left(\frac{1}{\mu} + \tau \right).$$

Additionally, we note that the Laplace transform of the *pdf* of the processing time distribution, denoted by $k^*(s)$, is

$$k^*(s) = \frac{\mu e^{-s\tau}}{\mu + s}. \quad (3)$$

3. Busy Period of a Single-server Queuing System with Delayed-Service

By a busy period of the system we mean an interval of time during which the system is continuously busy serving tasks. The stochastic law of the busy period for different queuing processes has attracted the attention of many researchers. Among methods used to find the distribution of the busy period are solving functional or difference equations, Rouche's Theorem, induction, solving combinatorial or integral equation. In this paper we will employ the functional equation method developed by Takács (1963) and utilize some of his results.

Let the random variable $\xi(t)$ denote the number of tasks in the system at time t , including the one being processed. We denote the CDF of a busy period, T_{bp} , by $B(t)$, i.e., $B(t) = P(T_{bp} \leq t)$. In general, it is possible that a busy period starts at the arrival of a task not necessarily to an idle system, for instance when there are $m - 1$, $m \geq 1$, tasks in the system, and ends the next time when the system returns to $m - 1$, i.e., the time interval $t_2 - t_1 \equiv T_{bp}$ between two consecutive instances t_1 and t_2 such that $\xi(t_1) - \xi(t_2) = m - 1$. This case is referred to an m -fold busy period. Thus, $\xi(t) \geq m, \forall t \in (t_1, t_2)$. We in this paper consider the case when $m = 1$.

Lemma 1: The n^{th} convolution of density function of $K(t)$, denoted by $k(t)$, with itself is

$$k_n(t) = \begin{cases} \frac{\mu^n (t - n\tau)^{n-1} e^{-\mu(t-n\tau)}}{(n-1)!}, & t \geq n\tau, n \geq 1 \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Proof: In probability theory, a convolution of two density functions $f(t)$ and $g(t)$ of two non-negative random variables, denoted by $(f * g)(t)$ is defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(u)g(t-u)du. \quad (5)$$

Note that $(f * g)(t) = (g * f)(t)$.

From (1), for $n = 1$ we have

$$k_1(t) = \begin{cases} \mu e^{-\mu(t-\tau)}, & t \geq \tau \\ 0, & \text{otherwise} \end{cases}.$$

For $n = 2$ from (4) we will get $k_2(t) = [k_1(t)] * [k_1(t)] = \int_{-\infty}^{\infty} k_1(u)k_1(t-u)du$. Using (1) we have

$$k_1(u) = \begin{cases} \mu e^{-\mu(u-\tau)}, & u \geq \tau \\ 0, & u < \tau \end{cases}, \quad \text{and } k_1(t-u) = \begin{cases} \mu e^{-\mu(t-u-\tau)}, & t-u \geq \tau \text{ or } u \leq t-\tau \\ 0, & t-u < \tau \text{ or } u > t-\tau \end{cases}.$$

$$k_2(t-u) = \int_{\tau}^{t-\tau} \mu e^{-\mu(u-\tau)} \mu e^{-\mu(t-u-\tau)} du = \begin{cases} \mu^2(t-2\tau)e^{-\mu(t-2\tau)}, & t \geq 2\tau \\ 0, & t > 2\tau \end{cases}.$$

Following the same approach, the proof can be completed by mathematical induction.

The next Lemma, Lemma 2, is from Takács (1962, p. 46), adjusted to our model.

Lemma 2. If $\text{Re}(s) \geq 0$, then $z = \gamma(s)$, the root of the equation

$$z = \frac{\mu e^{[s+\lambda(1-z)]\tau}}{s + \lambda(1-z) + \mu}, \tag{6}$$

which has the smallest absolute value, is

$$\gamma(s) = \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{j!} \int_0^{\infty} e^{-(\lambda+s)x} x^{j-1} dK_j(x), \tag{7}$$

where $K_j(x)$ denotes the j^{th} convolution of $K(x)$ with itself. if $\text{Re}(s) \geq 0$, $\gamma(s)$ is a continuous function of s . Further, if $\text{Re}(s) > 0$ or $\text{Re}(s) \geq 0$ and $\rho_Y > 1$, then $z = \gamma(s)$ is the only root of (6) within the unit circle $|z| < 1$. Specifically, $\omega = \gamma(0)$ is the smallest positive real root of the equation $\omega = \frac{\mu e^{\lambda(1-\omega)\tau}}{\lambda(1-\omega) + \mu}$. Moreover, if $\rho_Y > 1$, then $\omega < 1$ and if $\rho_Y \leq 1$, then $\omega = 1$.

Next we give the cumulative distribution function and mean of the busy period T_{bp} . It is clear from definition that the length of a busy period is independent of the queue discipline. Indeed this is the fact that Takács in (1962, p. 32) has “brilliantly” (as Kleinrock (1975, p. 210) puts it) used to find the distribution of a busy period.

Theorem 1:

- a. For a single-server Poisson arrival queue with delayed-service, if $\tau < \frac{1}{\lambda} - \frac{1}{\mu}$, the

cumulative distribution of the busy period T_{bp} is given by

$$B(t) = \sum_{n=1}^{\left\lfloor \frac{t}{\tau} \right\rfloor} \frac{\lambda^{n-1} \mu^n e^{\mu\tau n}}{n!} \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!(n-k-1)!} \frac{(-n\tau)^{n-k-1}}{(\lambda + \mu)^{n+k}} \times \sum_{i=0}^{n+k-1} \frac{1}{i!} \left[e^{-(\lambda+\mu)n\tau} [n\tau(\lambda + \mu)]^i - e^{-(\lambda+\mu)t} [t(\lambda + \mu)]^i \right], \tag{8}$$

where $\left\lfloor \frac{t}{\tau} \right\rfloor$ is the greatest integer less than or equal to $\frac{t}{\tau}$.

- b. $B(\infty) = 1$. (9)

c. $E(T_{bp}) = \begin{cases} \frac{1 + \mu\tau}{\mu - \lambda(1 + \mu\tau)}, & \text{if } \rho_Y < 1 \\ \infty, & \text{if } \rho_Y \geq 1 \end{cases}.$ (10)

Proof:

a. If we denote by $B^{(n)}(t)$ the probability that a busy period consists of n consecutive processing

and has a length $\leq t$, then $B(t) = \sum_{n=1}^{\infty} B^{(n)}(t)$. Now, whether the busy period starts when the

system is empty or in the case of m -fold busy period, with the arrival of the m^{th} task, the period ends when the first to be processed and all arrivals during its processing time complete their processing. This leads to the well-known functional equation $B^*(s) = K^*[s + \lambda - \lambda B^*(s)]$, where $K^*(\cdot)$ and $B^*(\cdot)$ are the Laplace-Stieltjes transform of $K(\cdot)$ and $B(\cdot)$, respectively, (see Kleinrock (1975, p. 212)). Moreover, Takács (1963) gives an explicit form for the busy period distribution for an $M/G/1$ as

$$B(t) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^t e^{-\lambda x} x^{n-1} dH_n(x), \quad (11)$$

where $H_n(x)$ is the n^{th} convolution of $H(x)$ (the service distribution) with itself. Note that $H(x)$ in our case is $K(x)$ (the processing time). From Lemma 1 and (11) we obtain that the distribution of T_{bp} for our model is given by:

$$B(t) = \sum_{n=1}^{\left[\frac{t}{\tau}\right]} \frac{\lambda^{n-1} \mu^n e^{\mu\tau n}}{n!} \int_{n\tau}^t \frac{x^{n-1} (x - n\tau)^{n-1} e^{-(\lambda+\mu)x}}{(n-1)!} dx \quad (12)$$

Using the Binomial theorem

$$(x - n\tau)^{n-1} = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} x^k (-n\tau)^{n-k-1},$$

(12) becomes

$$B(t) = \sum_{n=1}^{\left[\frac{t}{\tau}\right]} \frac{\lambda^{n-1} \mu^n e^{\mu\tau n}}{n!} \sum_{k=0}^{n-1} \frac{(-n\tau)^{n-k-1}}{k!(n-k-1)!} \int_{n\tau}^t x^{n+k-1} e^{-(\lambda+\mu)x} dx. \quad (13)$$

Integrating by parts in (13) and applying mathematical induction yields

$$\int_{n\tau}^t x^{n+k-1} e^{-(\lambda+\mu)x} dx = \frac{(n+k-1)!}{(\lambda+\mu)^{n+k}} \left\{ e^{-(\lambda+\mu)n\tau} \sum_{i=0}^{n+k-1} \frac{[n\tau(\lambda+\mu)]^i}{i!} - e^{-(\lambda+\mu)t} \sum_{i=0}^{n+k-1} \frac{[t(\lambda+\mu)]^i}{i!} \right\} \quad (14)$$

Therefore, from (13) and (14) we obtain (8).

b. It is clear that $\rho_Y = \lambda \left(\frac{1}{\mu} + \tau \right) \leq 1$ is equivalent to $\tau < \frac{1}{\lambda} - \frac{1}{\mu}$. From (11) we have $B(0) = 0$.

Now let $\mathcal{L}[B(t)] \equiv B^*(s)$ denote the Laplace transform of $B(t)$. It is well known that

$$\mathcal{L}\left(\frac{dB(t)}{dt}\right) = sB^*(s) \text{ and } \lim_{s \rightarrow 0} sB^*(s) = \lim_{s \rightarrow 0} \int_0^{\infty} e^{-\lambda s} B'(t) dt = \int_0^{\infty} B'(t) dt = \lim_{t \rightarrow \infty} B(t). \text{ Therefore,}$$

$$\lim_{t \rightarrow \infty} B(t) = \lim_{s \rightarrow 0} sB^*(s). \quad (15)$$

Hence, after some algebraic manipulations, using (11), and (7) we obtain

$$\begin{aligned}
 sB^*(s) &= \int_0^\infty B(t)se^{-\lambda s} dt = -\int_0^\infty B(t)d(e^{-\lambda s}) \\
 &= -\int_0^\infty \sum_{n=1}^\infty \frac{\lambda^{n-1}}{n!} \int_{n\tau}^t x^{n-1} e^{-\lambda x} dK_n(x) d(e^{-st}) \\
 &= -\sum_{n=1}^\infty \int_{n\tau}^\infty \frac{\lambda^{n-1} x^{n-1}}{n!} e^{-\lambda x} \int_x^\infty d(e^{-st}) dK_n(x) \\
 &= \sum_{n=1}^\infty \frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-(\lambda+s)x} x^{n-1} dK_n(x) = \gamma(s,1). \tag{16}
 \end{aligned}$$

Therefore, if $\lambda \left(\frac{1}{\mu} + \tau \right) \leq 1$, from (15), (16) and Lemma 2 we have,

$$\lim_{t \rightarrow \infty} B(t) = \lim_{s \rightarrow 0} \gamma(s) = \gamma(0) = \omega = 1, \text{ as stated in part (b) of the theorem.}$$

- c. To show (10), we use formula (47) of Theorem 6, p. 64, from Takács (1962). It states that expected length of a busy period is given by $\alpha/(1-\lambda\alpha)$ for $\lambda\alpha < 1$ and ∞ for $\lambda\alpha \geq 1$, where α is the mean service time. In our case $\alpha = \frac{1}{\mu} + \tau$ and direct substitution prove the result in part (c) of the theorem.

4. Queue Length Analysis

In what follows we aim to analyze the queue length for a single-server Poisson queue with delayed-service. It is well known that an Erlang distributed random variable with large number of stages and appropriately chosen mean value can be used to approximate a deterministic time. Therefore, the delay time τ can be approximated by an Erlang distributed random variable of order k , E_k , with large number of stages k and appropriate mean value (see Cox and Miller (1984) page 187). It is well-known that $E_k = X_1 + X_2 + \dots + X_k$, where $\{X_i\}$ is a sequence of iid

random variables with exponential distribution with parameter k/τ , i.e., $P\{X_i \leq t\} = 1 - e^{-\frac{k}{\tau}t}$,

with $E(X_i) = \tau/k$ and $Var(X_i) = \tau^2/k^2$. Thus, $E(E_k) = k \frac{\tau}{k} = \tau$, $Var(E_k) = k \frac{\tau^2}{k^2} = \frac{\tau^2}{k}$ and

$\lim_{k \rightarrow \infty} Var(E_k) = 0$. Therefore, we have $Y = \tau + X \approx \widehat{Y} = E_k + X$. In other words, tasks arrive in the system and eventually enter the processing station one at a time, where the processing requires passing through $k + 1$ stages. Times for these stages are $k + 1$ independent random variables, the first k of which are identically exponentially distributed with parameter k/τ and the last one is exponentially distributed with parameter μ .

Now using the random variable \widehat{Y} , the single-server Poisson queue with delayed-service can be approximated by $M/(E_k+X)/1$. Applying the method of stages, the functioning of $M/(E_k+X)/1$ can be described by a system of differential difference equations. In doing that, we denote the

state of the system by 0 when the system is empty and by the ordered pair (n, j) , where $n, n \geq 1$, is the number of tasks in the system and $j, 1 \leq j \leq k + 1$, is the stage of our processing. We will label the stages in the following order: the last (k^{th}) part of the Erlang portion as the 1st stage, ..., the first part of the Erlang portion as the k^{th} stage and the remaining exponential part (with parameter μ) of the processing as the $k+1^{st}$ stage of the processing time. Figure 1 shows these labeling.

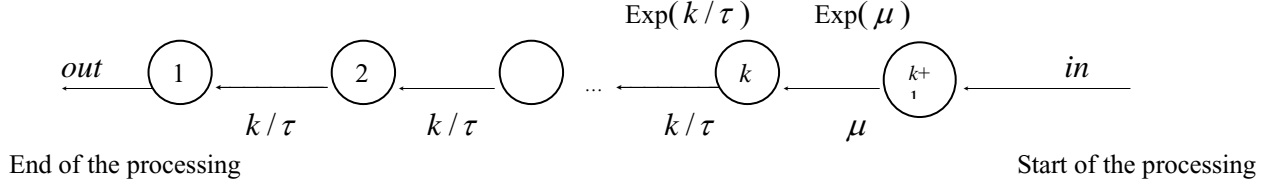


Figure 1. Stages of processing of a task

Note that a task cannot enter stage 1 until all stages are empty. Note also that the transition rate from each stage of the Erlang part to its neighboring stage during a delay period is k/τ , while transition rate from the exponential part during a service period is μ . The states of the system described by 0 and ordered pairs (n, j) are listed in Table 1.

Table 1. States of the System

State	Description
0	The empty state
$(1, i), i = 1, 2, \dots, k-1$	One task in a delay stage i
$(1, k)$	One task in the stage just before the service (the last stage of delay)
$(1, k+1)$	One task in the service stage
$(n, i), i = 1, 2, \dots, k-1$	n tasks in the system, one task is in a delay stage i
(n, k)	n tasks in the system, one task in the last stage of delay (just before the service)
$(n, k+1)$	n tasks in the system, one task in the service stage

Let the probability of $n \geq 0$ tasks in the system at time t be denoted by $P_n(t)$, i.e., $P_n(t) = P\{\xi(t) = n\}$. Further, let $P_n = \lim_{t \rightarrow \infty} P_n(t)$ be the steady-state probability of having n tasks in the system. Let $P_0(t)$ represent the probability that the system is empty and the joint distribution $P_{n,j}(t)$ represents the probability of n tasks in the system and one task in a processing stage $j, j = 1, 2, \dots, k-1, k, k+1$. Table 2 illustrates the possible transition of the states with their corresponding rates. Finally, let $P_{n,j} = \lim_{t \rightarrow \infty} P_{n,j}(t)$ be the steady-state probability of having n tasks in the system, and the processing is in state $j, j = 1, 2, \dots, k + 1$.

Table 2. Rates of Transition of States

Rate	From state	To state
λ	0	$(1, k+1)$
k/τ	$(1, 1)$	0
λ	$(n-1, i), \quad i = 1, 2, \dots, k-1$	$(n, i), \quad i = 1, 2, \dots, k-1, n \geq 2$

k/τ	$(1, i+1), \quad i = 1, 2, \dots, k-1$	$(1, i), \quad i = 1, 2, \dots, k-1$
λ	$(n-1, k), \quad n \geq 2$	$(n, k), \quad n \geq 2$
μ	$(1, k+1)$	$(1, k)$
λ	$(n-1, k+1), \quad n \geq 2$	$(n, k+1), \quad n \geq 2$
k/τ	$(2, 1)$	$(1, k+1)$
k/τ	$(n+1, 1), \quad n \geq 2$	$(n, k+1), \quad n \geq 2$
k/τ	$(n+1, i+1), \quad i = 1, 2, \dots, k-1, \quad n \geq 2$	$(n, i), \quad i = 1, 2, \dots, k-1, \quad n \geq 2$
μ	$(n+1, k+1), \quad n \geq 2$	$(n, k), \quad n \geq 2$

Thus, for a fixed k and a fixed $\tau > 0$, from Figure 1 and Table 2 we derive the following time-dependent system of differential difference equations describing the functioning of $M/(E_k+X)/1$ (see Gross and Harris (1998, p. 133)):

$$\begin{aligned}
P_0'(t) &= -\lambda P_0(t) + \frac{k}{\tau} P_{1,1}(t) \\
P_{1,i}'(t) &= -\left(\lambda + \frac{k}{\tau}\right) P_{1,i}(t) + \frac{k}{\tau} P_{1,i+1}(t), \quad 1 \leq i \leq k-1 \\
P_{1,k}'(t) &= -\left(\lambda + \frac{k}{\tau}\right) P_{1,k}(t) + \mu P_{1,k+1}(t) \tag{17}
\end{aligned}$$

$$\begin{aligned}
P_{1,k+1}'(t) &= -(\lambda + \mu) P_{1,k+1}(t) + \lambda P_0(t) + \frac{k}{\tau} P_{2,1}(t) \\
P_{n,i}'(t) &= -\left(\lambda + \frac{k}{\tau}\right) P_{n,i}(t) + \frac{k}{\tau} P_{n,i+1}(t) + \lambda P_{n-1,i}(t), \quad 1 \leq i \leq k-1, \quad n \geq 2 \\
P_{n,k}'(t) &= -\left(\lambda + \frac{k}{\tau}\right) P_{n,k}(t) + \mu P_{n,k+1}(t) + \lambda P_{n-1,k}(t), \quad n \geq 2 \\
P_{n,k+1}'(t) &= -(\lambda + \mu) P_{n,k+1}(t) + \frac{k}{\tau} P_{n+1,1}(t) + \lambda P_{n-1,k+1}(t), \quad n \geq 2
\end{aligned}$$

with

$$P_0(t) + \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} \sum_{j=1}^{k+1} P_{n,j}(t) = 1. \tag{18}$$

Note that the distribution of the number of tasks in the single-server Poisson queueing system with delayed-service at any time t will be

$$P_n(t) = \lim_{k \rightarrow \infty} \sum_{j=1}^{k+1} P_{n,j}(t). \tag{19}$$

4.1. Steady-State Distribution for the single-server Poisson queue with Delayed-Service

Applying Pollaczek-Khintchine formula (see Gross and Harris (1998, p. 212)), the mean queue length for the single-server Poisson queue with delayed-service, L_D , is

$$L_D = \frac{2\rho_Y(1-\lambda\tau) + \lambda^2\tau^2}{2(1-\rho_Y)}. \tag{20}$$

Later we will confirm (20) using the generating function of the steady-state distribution of the single-server Poisson queue with a delayed-service.

Our next goal is to solve for the distribution of $M/(E_k+X)/1$ and use it in order to obtain the steady-state for the delayed-service single-server Poisson queueing system. In order to solve the system (17), we define the following sub-generating functions $G_j(z)$, $1 \leq j \leq k+1$, and the generating function $G(z)$ of P_n , namely,

$$\begin{aligned} G_j(z) &= \sum_{n=1}^{\infty} P_{n,j} z^n, \quad 1 \leq j \leq k+1 \\ G(z) &= P_0 + \lim_{k \rightarrow \infty} \sum_{i=1}^{k+1} G_i(z). \end{aligned} \quad (21)$$

Theorem 2: If $\rho_Y < 1$, the steady-state solution of system (17) exists and it is given by

$$P_0 = 1 - \rho_Y. \quad (22)$$

$$P_n = (1 - \rho_Y) d_n, \quad n = 1, 2, \dots, \quad (23)$$

where

$$d_n = \begin{cases} \frac{(\lambda\tau)^{n-1}(\lambda\tau - n)}{n!} - \sum_{i=1}^n b_i d_{n-i}, & n = 1, 2, \dots, \\ 1, & n = 0 \end{cases}, \quad (24)$$

and

$$\begin{cases} b_1 = \lambda\tau - \left(1 + \frac{\lambda}{\mu}\right) e^{\lambda\tau}, \\ b_2 = \frac{(\lambda\tau)^2}{2} + \frac{\lambda}{\mu} e^{\lambda\tau}, \\ b_i = \frac{(\lambda\tau)^i}{i!}, \quad i = 3, 4, \dots. \end{cases} \quad (25)$$

Proof: From system (17) and equation (18) for the steady-state case the balance equations are

$$\begin{aligned} \lambda P_0 &= \frac{k}{\tau} P_{1,1} \\ \left(\lambda + \frac{k}{\tau}\right) P_{1,i} &= \frac{k}{\tau} P_{1,i+1}, \quad 1 \leq i \leq k-1 \\ \left(\lambda + \frac{k}{\tau}\right) P_{1,k} &= \mu P_{1,k+1} \\ (\lambda + \mu) P_{1,k+1} &= \lambda P_0 + \frac{k}{\tau} P_{2,1} \\ \left(\lambda + \frac{k}{\tau}\right) P_{n,i} &= \frac{k}{\tau} P_{n,i+1} + \lambda P_{n-1,i}, \quad 1 \leq i \leq k-1, \quad n \geq 2 \\ \left(\lambda + \frac{k}{\tau}\right) P_{n,k} &= \mu P_{n,k+1} + \lambda P_{n-1,k}, \quad n \geq 2 \end{aligned} \quad (26)$$

$$(\lambda + \mu)P_{n,k+1} = \frac{k}{\tau}P_{n+1,1} + \lambda P_{n-1,k+1}, \quad n \geq 2$$

with

$$P_0 + \lim_{t \rightarrow \infty} \sum_{n=0}^{\infty} \sum_{j=1}^{k+1} P_{n,j} = 1.$$

Keeping (19) in mind, by proper multiplication of the equations of (26) by powers of z , summing over appropriate indices, and some algebraic manipulation we obtain the following:

$$\left[\lambda(1-z) + \frac{k}{\tau} \right] G_i(z) - \frac{k}{\tau} G_{i+1}(z) = 0, \quad i = 1, 2, \dots, k-1 \quad (27)$$

$$\left[\lambda(1-z) + \frac{k}{\tau} \right] G_k(z) - \mu G_{k+1}(z) = 0 \quad (28)$$

$$[\lambda(1-z) + \mu] G_{k+1}(z) - \frac{k}{\tau z} G_1(z) = \lambda P_0(z-1) \quad (29)$$

From (27) - (29) obtain that

$$\begin{aligned} G_k(z) &= \left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^{k-1} \cdot G_1(z) \\ G_{k+1}(z) &= \frac{k}{\mu\tau} \left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^k \cdot G_1(z) \\ G_1(z) &= \frac{\lambda\mu\tau z(z-1)P_0}{k \left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^k \left\{ z[\lambda(1-z) + \mu] - \frac{\mu}{\left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^k} \right\}}. \end{aligned} \quad (30)$$

Since $\lim_{k \rightarrow \infty} \left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^k = e^{\lambda\tau(1-z)}$, from (30) we have $\lim_{k \rightarrow \infty} G_1(z) = 0$. However,

$$\lim_{k \rightarrow \infty} \left\{ 1 - \left[\frac{\lambda\tau}{k}(1-z) + 1 \right]^k \right\} \cdot \frac{G_1(z)}{\frac{\lambda\tau}{k}(z-1)} = \frac{\mu z (e^{\lambda\tau(z-1)} - 1) P_0}{z[\mu - \lambda(z-1)] - \mu e^{\lambda\tau(z-1)}}.$$

Thus, from (21) we express $G(z)$ as

$$G(z) = \left\{ \frac{\mu(z-1)e^{\lambda\tau(z-1)}}{-\lambda z(z-1) + \mu[z - e^{\lambda\tau(z-1)}]} \right\} P_0. \quad (31)$$

From (21) it is easy to see that $G(1) = 1$. Thus, from (31), using L'Hôpital's Rule we prove (22). Using (22), $G(z)$ can be rewritten as

$$G(z) = \frac{(\mu - \lambda - \lambda\mu\tau)(z-1)e^{\lambda\tau(z-1)}}{-\lambda z(z-1) + \mu[z - e^{\lambda\tau(z-1)}]}. \quad (32)$$

The probabilities P_n are the coefficients of z^n in $G(z)$. To obtain these coefficients, we write rewrite (32) as follows:

$$G(z) = \frac{1 + \sum_{n=1}^{\infty} \left[\frac{(\lambda\tau)^n}{n!} - \frac{(\lambda\tau)^{n-1}}{(n-1)!} \right] z^n}{1 + \left[\lambda\tau - \left(1 + \frac{\lambda}{\mu}\right) e^{\lambda\tau} \right] z + \left[\frac{(\lambda\tau)^2}{2} + \frac{\lambda}{\mu} e^{\lambda\tau} \right] z^2 + \sum_{n=3}^{\infty} \frac{(\lambda\tau)^n}{n!} z^n} \cdot P_0. \quad (33)$$

We now use the fact that the ratio of two power series is a power series and rewrite (33) in the following form:

$$G(z) = \frac{1 + \sum_{i=1}^{\infty} a_i z^i}{1 + \sum_{i=1}^{\infty} b_i z^i} P_0 = \sum_{i=0}^{\infty} (d_i P_0) z^i, \quad (34)$$

(see Gross and Harris (1998), page 218), where d_i and b_i , are given by (24) and (25), respectively. Hence, $G(z) = \sum_{n=0}^{\infty} (1 - \rho_Y) d_n z^n$, from which formula (23) follows. This completes the proof of Theorem 2.

Note that from (32) we have:

$$G'(1) = \frac{\lambda\mu(2 + 2\mu\tau - 2\lambda\tau - \lambda\mu\tau^2)}{2(\mu - \lambda - \lambda\mu\tau)^2} P_0. \quad (35)$$

Based on (22) and (35) we derive (20) using the generating function approach.

4.2. Transient Probability Distribution for a Single-server Queue with Delayed-Service

Similar to the steady-state case, we define the following sub-generating functions $G_j(z, t)$ for $1 \leq j \leq k+1$ and the generating function $G(z, t)$ of $P_n(t)$, namely,

$$G_j(z, t) = \sum_{n=1}^{\infty} P_{n,j}(t) z^n, \quad 1 \leq j \leq k+1 \quad (36)$$

$$G(z, t) = P_0(t) + \lim_{k \rightarrow \infty} \sum_{i=1}^{k+1} G_i(z, t) = P_0(t) + \lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} P_{n,k}(t) z^n \quad (37)$$

The following theorem holds:

Theorem 3: For the time-dependent solution of system (17) we have:

$$P_0^*(s) = \frac{1}{s + \lambda(1 - z_1)}, \quad (38)$$

and

$$G^*(z, s) = \frac{z + \mu z \left(\frac{1 - e^{-[s + \lambda(1-z)]\tau}}{s + \lambda(1-z)} \right) - \mu(1-z) \left(\frac{e^{-[s + \lambda(1-z)]\tau}}{s + \lambda(1-z_1)} \right)}{z[s + \mu + \lambda(1-z)] - \mu e^{-[s + \lambda(1-z)]\tau}}, \quad (39)$$

where z_1 is the root of the denominator of (39) within the unit circle.

Proof: We apply the same method we used for the steady-state case. However, we will also apply Laplace transform on the generating functions. Hence, we let $G_j^*(z, s)$ and $G^*(z, s)$ denote the Laplace transforms of $G_j(z, t)$ and $G(z, t)$, respectively. Therefore, from (17), (36) and (37) we will have:

$$G_{i+1}^*(z, s) = \left[\frac{\tau s}{k} + \frac{\lambda \tau}{k} (1-z) + 1 \right] G_i^*(z, s), \quad i = 1, 2, \dots, k-1, \quad (40)$$

$$\frac{k}{\mu \tau} \left[\frac{\tau s}{k} + \frac{\lambda \tau}{k} (1-z) + 1 \right] G_k^*(z, s) = G_{k+1}^*(z, s), \quad (41)$$

and

$$[s + \lambda(1-z) + \mu] G_{k+1}^*(z, s) - \frac{k}{\tau z} G_1^*(z, s) = [\lambda(1-z) - s] P_0^*(z, s) + 1. \quad (42)$$

Using (40) – (42) we obtain

$$G_k^*(z, s) = \left[\frac{\tau s}{k} + \frac{\lambda \tau}{k} (1-z) + 1 \right]^{k-1} G_1^*(z, s),$$

$$G_{k+1}^*(z, s) = \frac{k}{\mu \tau} \left[\frac{\tau s}{k} + \frac{\lambda \tau}{k} (1-z) + 1 \right]^k G_1^*(z, s),$$

and

$$G_1^*(z, s) = \frac{[\lambda \mu \tau z(z-1) - \mu \tau s z] P_0^*(z, s) + \mu \tau z}{\left\{ z[s + \lambda(1-z) + \mu] \left[\frac{\tau s}{k} + \frac{\lambda \tau}{k} (1-z) + 1 \right]^k - \mu \right\} k}. \quad (43)$$

It is easy to see that $\lim_{k \rightarrow \infty} \left[1 + \frac{\tau}{k} [s + \lambda(1-z)] \right] = 1$. Moreover, it is well known that

$\lim_{k \rightarrow \infty} \left[1 + \frac{\tau}{k} [s + \lambda(1-z)] \right]^k = e^{(s+\lambda)(1-z)\tau}$. Thus, from (43) we have $\lim_{k \rightarrow \infty} G_1^*(z, s) = 0$. From (37) we

can write $G^*(z, s)$ as

$$G^*(z, t) = \lim_{k \rightarrow \infty} \left[P_0^*(t) + \sum_{i=1}^{k+1} G_i^*(z, t) \right].$$

On the other hand, $P_0^*(t) + \sum_{i=1}^{k+1} G_i^*(z, t)$ can be found as

$$P_0^*(t) - \frac{k G_1^*(z, s)}{\tau [\lambda(1-z) + s]} \left\{ 1 - \left[1 + \frac{s + \lambda(1-z)}{k} \tau \right]^k \right\} + \frac{k G_1^*(z, s)}{\tau \mu} \left[1 + \frac{s + \lambda(1-z)}{k} \tau \right]^k. \quad (44)$$

Now taking limit of (44) as k approaches infinity, after simplification, we will have

$$\begin{aligned}
G^*(z, s) = P_0^*(s) &- \frac{\mu z \left[1 - e^{-[s+\lambda(1-z)]\tau} \right] P_0^*(s)}{z[s + \mu + \lambda(1-z)] - \mu e^{-[s+\lambda(1-z)]\tau}} \\
&+ \frac{\mu z \left[1 - e^{-[s+\lambda(1-z)]\tau} \right]}{[s + \lambda(1-z)] \left\{ z[s + \mu + \lambda(1-z)] + \mu e^{-[s+\lambda(1-z)]\tau} \right\}} \\
&+ \frac{z \left\{ 1 - [s + \lambda(1-z)] P_0^*(s) \right\}}{z[s + \lambda(1-z) + \mu] - \mu e^{-[s+\lambda(1-z)]\tau}},
\end{aligned}$$

or

$$G^*(z, s) = \frac{\mu z \left(\frac{1 - e^{-[s+\lambda(1-z)]\tau}}{s + \lambda(1-z)} \right) + z + \mu(1-z)e^{-[s+\lambda(1-z)]\tau} P_0^*(s)}{z[s + \mu + \lambda(1-z)] - \mu e^{-[s+\lambda(1-z)]\tau}}. \quad (45)$$

Note that for $z = 1$, from (45), we will have $G^*(s) = 1/s$.

To find $P_0^*(s)$, we note that the convergence of $G^*(z, s)$ implies that the roots of the denominator of (45) within the unit circle should also be roots of the numerator of (45). From Lemma 2, the denominator of (45) has a unique root in the unit circle. Let z_1 be the root described in Lemma 1, i.e., $|z_1| < 1$. Thus, the numerator of (45) will be zero at z_1 . Since $\text{Re}[s + \mu + \lambda(1-z_1)] > 0$ we will have (38). This leads to (39) and it completes the proof.

Similar to (38) result can be found in Takács (1962, p. 55).

5. Numerical Example

The stability condition $\tau < \frac{1}{\lambda} - \frac{1}{\mu}$, which is equivalent to $\rho_Y < 1$, is required to ensure that the steady-state distribution of the busy period exists. Furthermore, often for practical purposes, the delay time τ has to be relatively small. However, small value of τ with time units t greater than one, make $[t/\tau]$ to be very large. For instance, to evaluate the distribution of the busy period, when $t = 10$, and $\tau = 0.1$, would require nearly two million rounds of operations involving expressions such as 100^{100} , e^{100} and $100!$. Our experience shows that the mathematical software Matlab that we have used for our programming can handle a limited values of $[t/\tau]$, which will impose some constraints on the choice of λ , μ and τ . On the other hand, a large delay time will require a small arrival and large service rates for the steady-state to exist. This may not be desirable and, indeed, practically not acceptable. Based on the above constraints we have selected the values of λ , μ and τ for our examples. As noticed from Figures 2 and 3, the busy period probabilities are approaching 1 as time becomes large.

We have also illustrated comparative figures to show difference between the busy period of our model (on the graph denoted by $B(t)$), and the one for the standard $M/M/1$ (on the graph denoted

by $BMM(t)$, Figures 4 and 5. As it can be seen for values of τ very close to zero, $M/M/1$ is a good approximation for our model. It is our conjecture (apparent from our observations) that there exists a threshold for τ , say τ^* , such that if $\tau > \tau^*$ the two systems are significantly different. The difference between the two systems is identified by a prespecified performance diagnostics. In our Figures 4 and 5, for instance, .09 and .08 are closed to the threshold.

For the queue length distribution, Figures 6 – 9, cumulative probabilities are approaching 1, as expected. The size of τ in this case is of no concern.

6. Conclusion/Discussion:

In this paper we discuss a delayed service queueing model for the first time. We prove that it is a non-Markovian and set it up as an $M/G/1$ queueing model. This way we are able to apply some of the available for $M/G/1$ results in the literature. We provide an explicit closed form distribution for the length of a busy period. The delay time τ is approximated by an Erlang distributed random variable with a large number of stages. Thus, we are able to write the system of differential difference equations for the new model and obtain the distribution of the queue length in both steady-state and transient cases. An explicit closed form steady-state distribution for the queue length is obtained. Additionally not only we obtain the Laplace transform of the generating function of the transient queue length, but also, we derive the Laplace transform of the probability of the system to be empty.

For the numerical examples, we use Matlab. We choose different values of the parameters λ , μ , τ , and t to examine the behavior of the distribution of the busy period as well as the distribution of the queue length given in Figures 4 – 9. Based on the numerical examples, we observe that when the values of τ is “very close” to its upper limit $\frac{1}{\lambda} - \frac{1}{\mu}$, the distribution of the queue length converges very slowly to one.

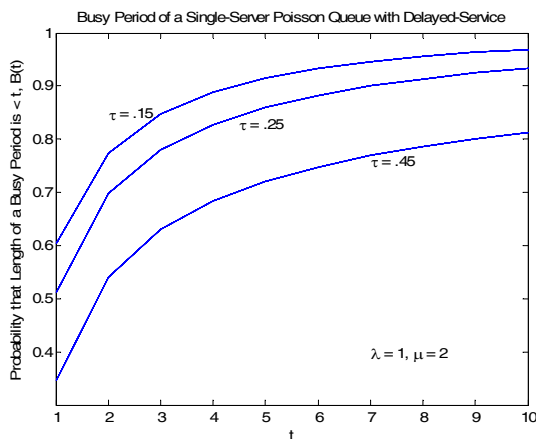


Figure 2

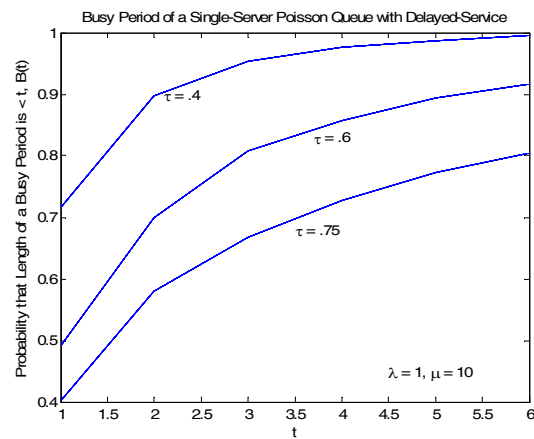


Figure 3

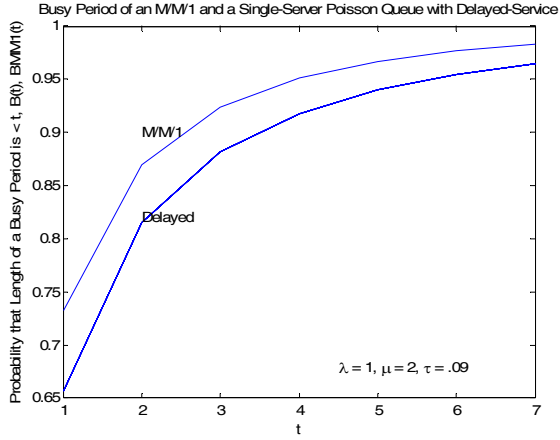


Figure 4

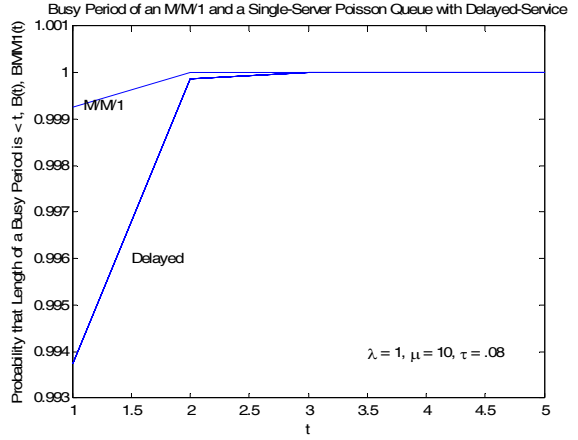


Figure 5

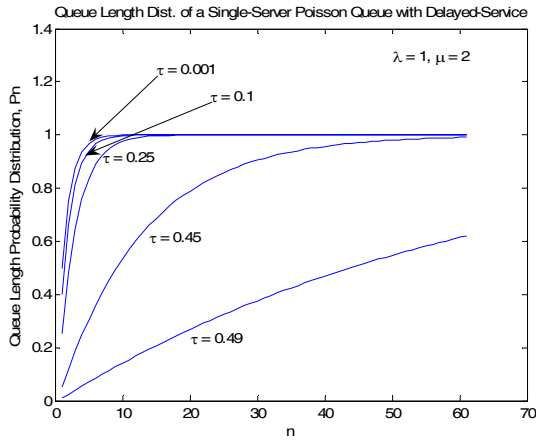


Figure 6

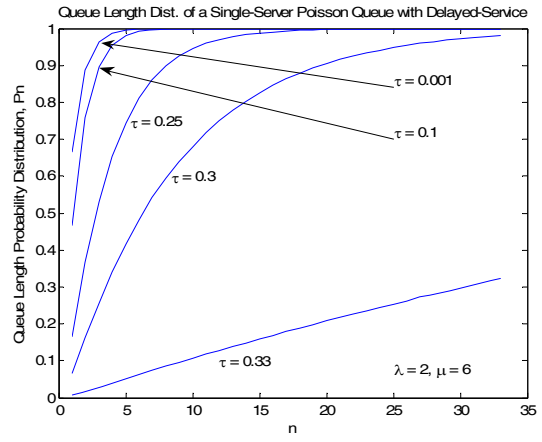


Figure 7

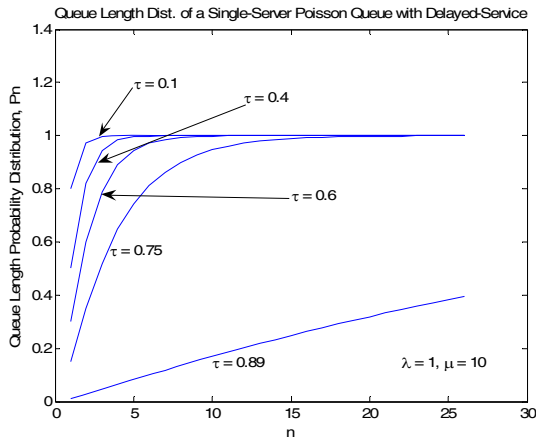


Figure 8

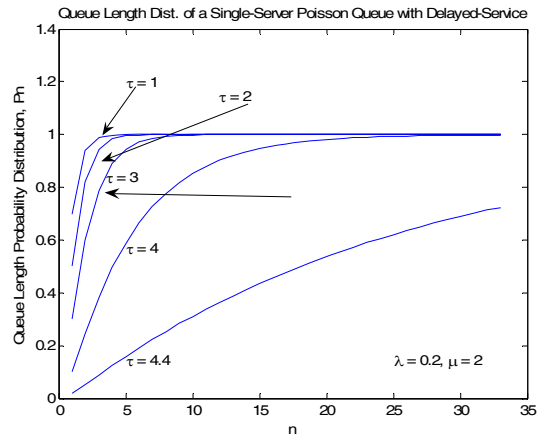


Figure 9

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