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The dynamics of asset lifetime under technological change

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ABSTRACT

The variable lifetime of assets is analyzed in a serial replacement problem. Technological change impacts the maintenance cost and new asset cost. The optimal asset lifetime appears to be constant *only* when both costs decrease with the same rate. We identify cases when the technological change decreases or increases the optimal lifetime.

1. Introduction

First optimal asset replacement models were developed by R.Bellman in 1955 for the variable lifetime of assets [1]. In 1976, E.Elton and M.Gruber proved that an equal life policy was optimal on the infinite horizon under *technological change* (TC) [6]. Correspondingly, many later replacement models assumed that the optimal lifetime of assets was constant. However, it does not hold in more general cases of TC [3,9–14]. The TC impacts both the operating and maintenance (O&M) cost and the purchase cost of possible replacement assets [2,14,15]. This paper identifies cases when the optimal asset lifetime is variable or constant and when the TC increases or decreases the optimal lifetime.

An important open question is how TC impacts the optimal lifetime of assets. The majority of authors conclude that the optimal asset lifetime is shorter under more intensive TC [2,7,11, 14,15], although the opposite view also exists [3,5]. This paper explains why it happens and demonstrates that both situations are possible (which concludes the discussion started in [5,11]).

The replacement models represent *serial replacement* of a single asset or *parallel replacement* of many economically dependent assets. [2,3,5,7,11,14,15].

The next section describes a serial replacement model over the infinite continuous-time horizon with taking salvage values into account. Section 3 derives the extremum condition for this model. Section 4 identifies the case when the optimal lifetime of assets is constant and shows that it happens *if and only if* the TC rate is the same for both O&M and new asset costs (a "proportional" TC). The optimal lifetime is always shorter for more intense proportional TC. Section 5 explores a general case and shows that the optimal asset lifetime increases if the O&M cost decreases faster than the new asset cost, and converse. The TC impact on the optimal lifetime appears to depend on the cost component influenced by the TC: the optimal lifetime is shorter under more intense TC in the new asset cost and is longer under a faster decrease of the O&M cost.

2. Model and optimization problem

We consider the replacement process of a single asset in the continuous time t, $t \in [\tau_0, \infty)$, assuming that the initial purchase time τ_0 of the asset is given. The *replacement policy* π is the sequence $\{L_i, i = 1, 2, \ldots,\}$ of the unknown lifetimes L_i of the consecutively replaced asset. This sequence directly determines the sequence $\{\tau_i, i = 1, 2, \ldots,\}$ of the replacement times $\tau_i = L_i + \tau_{i-1}$.

The present value of the total cost of the replacement policy over the infinite horizon $[\tau_0, \infty)$ can be expressed as

$$J(\tau_1, \tau_2, \ldots) = \sum_{i=1}^{\infty} \mathrm{e}^{-r\tau_i} p(\tau_i) + \sum_{i=0}^{\infty} \left[\int_{\tau_i}^{\tau_{i+1}} \mathrm{e}^{-ru} q(\tau_i, u) \mathrm{d}u - \sigma \mathrm{e}^{-r\tau_{i+1}} p(\tau_i) \mathrm{e}^{-s(\tau_{i+1}-\tau_i)} \right], \quad (1)$$

where $p(t), t \in [\tau_0, \infty)$, is the cost of a new asset (purchase price and installation cost) at time t; $q(t, u), t, u \in [\tau_0, \infty)$, is the operating and maintenance (O&M) rate at the time u for the

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asset bought at time $t \le u$; $\sigma, 0 \le \sigma < 1$, is the salvage value multiplier for the new asset; $s > 0$ is the instantaneous decrease rate of the salvage value; and $r > 0$ is the instantaneous discount rate. Because of deterioration, the O&M cost $q(t, u)$ increases in u at fixed t as the asset becomes older (the asset age $u - t$ increases). The TC leads to the availability of newer assets (challengers) that require less maintenance and are less expensive, i.e. $p(t)$ and $q(t, u)$ decrease in t for any fixed asset age $u - t$.

In (1), the first term represents the total price of purchased assets, the second term is the discounted total O&M cost, an the third one is the discounted total salvage value [7,14,15].

We formulate the replacement problem as finding the optimal policy $\pi^* = \{\tau_i^*, i = 1, 2, \ldots\}$ that minimizes the replacement cost (1):

$$J(\tau_1^*, \tau_2^*, \ldots) = \min_{\tau_i, i=1,\ldots,\infty} J(\tau_1, \tau_2, \ldots). \tag{2}$$

Model (1) and (2) covers several known models [7,14,15] and others. Model [14] considers an analogous optimization problem in the *discrete time* under the assumption of geometric TC and deterioration, while [7,15] assume that $p(i) = \text{const}$, $q(i, k)$ is linear in $k - i$, and $\tau_{i+1} - \tau_i = L_i = \text{const}$, $i = 1, 2, \ldots$. The models [14,15] assume that the unknowns τ_i are integer-valued, which essentially complicates the analysis. Then (1)–(2) is an integer programming problem and no tools are available for its qualitative analysis. In line with [7], we consider the continuous model (1) for the positive real-valued τ_i^*. Then it can be investigated using standard optimization techniques.

3. Extremum condition

Although the minimized function (1) is expressed through the unknowns τ_i, $i = 1, 2, \ldots$, the extremum condition appears to have a simpler form in the unknowns $L_i = \tau_i - \tau_{i-1}$, $i = 1, 2, \ldots$.

Lemma 1 (*The Necessary Condition for an Extremum*). *If an optimal policy $\pi^* = \{L_i^*, i = 1, 2, \ldots\}$ exists, then every unknown L_i^*, $0 < L_i^* < \infty$, satisfies the condition*

$$\begin{aligned} &-rp(\tau_i) + p'(\tau_i)\left[1 - \sigma e^{-(r+s)L_{i+1}}\right] + \sigma\left[(r+s)p(\tau_{i-1})e^{-sL_i}\right.\\ &\quad\left. - sp(\tau_i)e^{-(r+s)L_{i+1}}\right] + q(\tau_{i-1}, \tau_i) - q(\tau_i, \tau_i)\\ &\quad + \int_{\tau_i}^{\tau_{i+1}} e^{-r(u-\tau_i)} \frac{\partial q(\tau_i, u)}{\partial \tau_i} du = 0, \quad i = 1, 2, \ldots. \end{aligned} \tag{3}$$

Proof. The necessary condition for an extremum of the function $J(\tau_1, \tau_2, \tau_3, \ldots)$ is

$$\partial J/\partial \tau_i = 0, \quad i = 1, 2, \ldots. \tag{4}$$

For a fixed i, the unknown variable τ_i appears in five terms of the objective function (1). So, we can present (1) as a (finite) function of τ_i and an infinite sum that does not depend on τ_i. Differentiating (1) in τ_i, we obtain the expression

$$\begin{aligned} \frac{\partial J}{\partial \tau_i} &= -re^{-r\tau_i}p(\tau_i) + e^{-r\tau_i}p'(\tau_i) + e^{-r\tau_i}q(\tau_{i-1}, \tau_i)\\ &\quad - e^{-r\tau_i}q(\tau_i, \tau_i) + \int_{\tau_i}^{\tau_{i+1}} e^{-ru}\frac{\partial q(\tau_i, u)}{\partial \tau_i} du\\ &\quad - \sigma p'(\tau_i)e^{-r\tau_{i+1} - s(\tau_{i+1} - \tau_i)} + \sigma(r+s)p(\tau_{i-1})e^{-r\tau_i - s(\tau_i - \tau_{i-1})}\\ &\quad - \sigma sp(\tau_i)e^{-r\tau_{i+1} - s(\tau_{i+1} - \tau_i)} = 0, \end{aligned}$$

which after transformation leads to equality (3). □

Let us impose specific assumptions on the given functions to provide a qualitative analysis of the optimal lifetime. Namely, we assume here and thereafter that both TC and deterioration are exponential:

$$\begin{aligned} &q(t, u) = q_0 e^{c_d(u-t)} e^{-c_q t}, \qquad p(t) = p_0 e^{-c_p t},\\ &c_q + c_d > 0, \quad 0 \leqslant c_p + c_d < r. \end{aligned} \tag{5}$$

Usually, both new asset and O&M costs decrease: $c_p > 0$, $c_q > 0$, because of TC. The TC impact on these costs can be different: $c_p \neq c_q$. By (5), the O&M and asset costs can even increase ($c_p < 0$ and/or $c_q < 0$) but slower than the deterioration rate c_d.

Under the exponential TC and deterioration (5), the extremum condition (3) has the form:

$$\begin{aligned} &\left[e^{(c_d+c_q)L_i} - 1\right] + \frac{c_d + c_q}{r - c_d}\left[e^{-(r-c_d)L_{i+1}} - 1\right]\\ &\quad + \sigma\frac{p_0}{q_0}e^{(c_q-c_p)\tau_i}\left[(c_p - s)e^{-(r+s)L_{i+1}} + (r+s)e^{(c_p-s)L_i}\right]\\ &= \frac{p_0}{q_0}(r + c_p)e^{(c_q-c_p)\tau_i}. \end{aligned} \tag{6}$$

4. Case of constant optimal asset lifetime (Proportional TC)

It is of obvious interest to identify possible cases of a constant optimal lifetime.

Definition. The TC *with the equal rates c_q and c_p* is referred to as *the proportional TC*.

In the proportional TC case $c_q = c_p$, equality (6) does not depend on τ_i explicitly. So, we can try the constant optimal lifetime $L_i \equiv L$ as a solution to (6).

Theorem 1. *If $c_q = c_p = c$ in (5), then the optimization problem (1)–(2) possesses a unique optimal policy $\pi^* = \{L_i^*, i = 1, 2, \ldots\}$ such that $L_i^* = L$, $i = 1, 2, \ldots$, where the constant $L > 0$ is uniquely determined from the non-linear equation*

$$\begin{aligned} &e^{(c+c_d)L} - 1 + \frac{c + c_d}{r - c_d}\left[e^{-(r-c_d)L} - 1\right] + \sigma\frac{p_0}{q_0}\left[(c - s)e^{-(r+s)L}\right.\\ &\quad\left. + (r+s)e^{(c-s)L}\right] - \frac{p_0}{q_0}(r + c) = 0. \end{aligned} \tag{7}$$

At small c, s, and r,

$$L \approx \left[\frac{2(1-\sigma)}{(c + c_d)q_0/p_0 + \sigma(c - s)(r + s)}\right]^{1/2}. \tag{8}$$

Proof. Substituting $L_i \equiv L$ to (6), we obtain (7). Let $F(L)$ denote the left side of (7), then $F(0) = (\sigma - 1)p_0(c + r)/q_0 < 0$. The behavior of $F(L)$ at large L is determined by two exponents with the positive coefficients $(c + c_d)L$ and $(c - s)L$, hence, $F(L) \to \infty$ at $L \to \infty$. Since $F(L)$ is continuous, the Eq. (7) has a solution $L^* > 0$. Finally, the derivative

$$\begin{aligned} \frac{dF}{dL} &= (c + c_d)\left[e^{(c+c_d)L} - e^{-(r-c_d)L}\right]\\ &\quad + \sigma\frac{p_0}{q_0}(r+s)(c-s)\left[e^{(c-s)L} - e^{-(r+s)L}\right] \end{aligned} \tag{9}$$

is positive under (5), hence, the solution L^* is unique. The theorem is proven. □

Therefore, in the case of *the proportional TC*, $c_q = c_p = c$, the optimal asset lifetime is constant.

Theorem 2. *Under the proportional TC $c_q = c_p = c$ and $\sigma = 0$, the (constant) optimal lifetime L^* is shorter when the TC rate c is larger.*

Proof. Let us consider (7) as the implicit function $F(L, c) = 0$. The derivative

$$\frac{\partial F}{\partial c} = Le^{(c+c_d)L} - \frac{p_0}{q_0} + \frac{1}{r - c_d}\left[e^{(c+c_d)L} - 1\right] + \sigma\frac{p_0}{q_0}\left[e^{-(r+s)L} + (r+s)Le^{(c-s)L}\right]$$

is $Le^{(c+c_d)L} - \left[e^{(c+c_d)L} - 1\right]/(c + c_d) + \frac{p_0}{q_0}\frac{r-c_d}{c+c_d} > 0$ at $L^* > 0$ and $\sigma = 0$. Next, $\frac{\partial F}{\partial L} > 0$ by (9) and $\frac{\partial L}{\partial c} = -\frac{\partial F}{\partial c}/\frac{\partial F}{\partial L} < 0$ by the theorem on the implicit function. The theorem is proven. □

5. Case of variable optimal asset lifetime

Under the exponential TC and deterioration (5), the constant optimal lifetime can appear only in the case of the proportional TC $c_q = c_p$. Indeed, if $c_q \neq c_p$, then the left side of Eq. (6) for L_i increases or decreases in τ_i (and in i). Therefore, the solution L_i (if exists) also depends on i.

The general case $c_q \neq c_p$ is more challenging. To analyze it, we assume the salvage value $\sigma = 0$. Then, the extremum condition (6) has the form:

$$\left[e^{(c_d+c_q)L_i} - 1\right] + \frac{c_d + c_q}{r - c_d}\left[e^{-(r-c_d)L_{i+1}} - 1\right] = \frac{p_0}{q_0}(r + c_p)e^{(c_q-c_p)\tau_i}, \quad i = 1, 2, \ldots. \tag{10}$$

Analogously to [11,13,16], Eq. (10) can be analyzed by means of integral equations.

Lemma 2. *If an optimal policy $\pi^* = \{L_i^*, k = 1, 2, \ldots\}$ exists, then*

$$L_i^* = \tau_i - x(\tau_i) \quad at\ \tau_i = \tau_{i-1} + L_i^*, i = 1, 2, \ldots. \tag{11}$$

where $x(t)$, $x(t) < t$, $t \in [\tau_0, \infty)$, is a unique monotonic *solution of the nonlinear integral equation*

$$q_0\int_t^{x^{-1}(t)} e^{-ru}[e^{-c_q x(u)}e^{c_d(u-x(u))} - e^{-c_q t}e^{c_d(u-t)}]du = p_0e^{-rt}e^{-c_p t}, t \in [\tau_0, \infty), \tag{12}$$

$x^{-1}(t)$ is the inverse function of $x(t)$.

Proof. Let us define $L(\tau_i) = \tau_i - \tau_{i-1}$, then $\tau_{i-1} = \tau_i - L(\tau_i)$ and $\tau_i = \tau_{i+1} - L(\tau_{i+1})$. Next, formally denoting $t = \tau_i$ in (10) and introducing the function $x(t) = t - L(t)$, we obtain $t = \tau_{i+1} - L(\tau_{i+1}) = x(\tau_{i+1})$ and $\tau_{i+1} = x^{-1}(t)$. Hence, the expression (10) can be rewritten as

$$\left[e^{(c_d+c_q)L(t)} - 1\right] + \frac{c_d + c_q}{r - c_d}\left[e^{-(r-c_d)(x^{-1}(t)-t)} - 1\right] = \frac{p_0}{q_0}(r + c_p)e^{(c_q-c_p)t}. \tag{13}$$

Alternatively, the differentiation of Eq. (12) and routine transformations lead to

$$q_0e^{-c_q t}\left[e^{(c_d+c_q)(t-x(t))} - 1\right] + q_0e^{-c_q t}\frac{c_d + c_q}{r - c_d} \times \left[e^{-(r-c_d)(x^{-1}(t)-t)} - 1\right] = p_0(r + c_p)e^{-c_p t}, \tag{14}$$

which is equivalent to (13). Hence, if (12) and, therefore, (14) hold for any $t \in [\tau_0, \infty)$, then the necessary extremum condition (10) also holds for the specific instants τ_k, $k = 1, 2, \ldots$.

The lemma is proven. □

By Lemma 2, if we know a solution to Eq. (12), we can find the optimal replacement lifetimes L_k^*, $k = 1, 2, \ldots$, for *any* given initial purchase time $\tau_0 < 0$.

In economic theory, similar replacement processes are described by *vintage capital models* with endogenous capital lifetime [4,8,9,12,13]. From the OR viewpoint, these models are the models of parallel replacement under TC. A parallel asset replacement model similar to (1)–(2), (5) was analyzed in [11,13,16] and also lead to Eq. (12) for optimal lifetime. This model [11] described replacement when the costs p and q did not depend on the number of simultaneously replaced assets (no economies of scale [3]). The parallel asset replacement is more complicated in the presence of adjustment costs and other economy-of-scale effects. Therefore, the Eq. (12) allows us to analyze the qualitative dynamics of the *variable* optimal asset lifetime in both serial (1)–(2) and parallel replacement models. In particular, Lemma 2 and the results of [16] produce the following property.

Theorem 3. *Under the exponential TC and deterioration (5) and $c_q \neq c_p$, the optimization problem (1)–(2) possesses a unique optimal policy $\pi^* = \{L_i^*, i = 1, 2, \ldots\}$ such that:*

(a) *If $c_p > c_q$, then the optimal lifetime decreases, $L_{i+1}^* < L_i^*$, $i = 1, 2, \ldots$, and L_i^* strives to 0 as $i \to \infty$.*
(b) *If $c_d < c_p < c_q$, then $L_{i+1}^* > L_i^*$, $i = 1, 2, \ldots$, and $L_{i+1}^* \cong L_i^*(c_q + c_d)/(c_p + c_d)$ as $i \to \infty$.*

Proof. The proof expands the technique suggested in [16] for the integral equation (12) without deterioration, that is, at $c_d = 0$. Namely, by Theorem 3 of [16], the Eq. (14) has a unique solution $x(t)$, $t \in [0, \infty)$, such that $x(t) \to t$ as $t \to \infty$. It proves case (a).

If $c_d < c_p < c_q$ (case (b)), then expression (14) at $t \to \infty$ produces

$$e^{(c_q+c_d)L(t)} = \frac{p_0}{q_0}(c_p + r)e^{(c_q-c_p)t} + o(e^{(c_q-c_p)t}). \tag{15}$$

Taking the logarithm of the both sides of (15), we obtain that

$$L(t) = \frac{c_q - c_p}{c_q + c_d}t + o(t) \approx \frac{c_q - c_p}{c_q + c_d}t \quad \text{at } t \gg 1. \tag{16}$$

Using $x(t) = t - L(t)$, we have $x(t) \approx (\frac{c_p+c_d}{c_q+c_d})t$ and $x^{-1}(t) \approx (\frac{c_q+c_d}{c_p+c_d})t$ at $t \to \infty$. Hence, if $L_i = L(t)$, then $L_{i+1} = L(x^{-1}(t)) \approx \frac{c_q-c_p}{c_p+c_d}t$. Combining the last equality and (16), we obtain $L_{i+1} = \frac{c_q+c_d}{c_p+c_d}L_i$, that proves the property (b). The theorem is proven. □

In conclusion, we provide the following property.

Theorem 4. *If $c_p^1 > c_p^2$, then $L_k^1 < L_k^2$ for all $k = 1, 2, \ldots$, where the optimal policies $\{L_k^1\}$ and $\{L_k^2\}$ correspond to c_p^1 and c_p^2. Alternatively, if $c_q^1 > c_q^2$, then $L_k^1 > L_k^2$, $k = 1, 2, \ldots$, where $\{L_k^1\}$ and $\{L_k^2\}$ correspond to c_q^1 and c_q^2.*

The proof is obtained by giving a small variation δc to the coefficient c_p or c_q and a small variation $\delta x(t)$, $t \in [\tau_0, \infty)$, to the solution $x(t)$ of Eq. (12) and investigating the dynamics of the obtained linear integral equation for $\delta x(t)$. It is pretty straightforward but requires tedious routine transformations, which we skip because of space limitations. □

Theorem 4 identifies cases when TC decreases or increases the optimal lifetime. Indeed, the optimal lifetimes are *shorter* when the TC in new asset cost is more intense (the rate c_p is larger) but the rate c_q in O&M cost remains the same. Hence, an acceleration of the TC in the new asset cost speeds up the introduction of new technologies. For the same new asset cost rate c_p, the optimal lifetimes are *longer* when the O&M cost rate c_q increases. Hence, a more intense TC in the O&M cost delays the introduction of new assets. The last property includes the case referred in [5] as a paradox in equipment replacement.

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