



Integral equation of optimal replacement: Analysis and algorithms

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ABSTRACT

The paper is devoted to theoretic and numeric investigation of a nonlinear integral equation with an unknown function in the upper limit of integration. This equation appears in optimal replacement problems of engineering and production economics. Its solution is essential for finding the optimal policy of equipment replacement under technological advances. Solvability and qualitative dynamics of the solution are analyzed. Computational algorithm for solving the equation is constructed. A numeric example illustrates the obtained results.

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1. Introduction

The optimal replacement of *machines (equipment, productive capital)* is one of the fundamental issues of engineering economics, operations research, and management sciences. The paper considers a simple process of the optimal replacement of one machine in continuous time t . To keep the machine indefinitely, a firm periodically sells the old machine and buys a new one. The machine is characterized by the purchase price $p(t)$ and the efficiency $b(t, u)$ of the machine bought at time t and used at time u , $t \leq u$. In the presence of technological change, newer machines are more efficient (productive) than the older ones. The optimal replacement process can be characterized by the installation time $x(t) < t$ of the machine that is being replaced at time t (known in [1,2] as the *scrapping time*). Then, $t - x(t)$ is the machine lifetime. In this paper, we show that the optimal replacement is described by the following nonlinear integral equation

$$\int_t^{x^{-1}(t)} e^{-ru} [b(t, u) - b(x(u), u)] du = e^{-rt} p(t), \quad t \in (-\infty, \infty), \quad (1)$$

with respect to x . The constant $r > 0$ is the industry-wide discount rate. The unknown x in the integrand and the upper limit of integration essentially complicates the analysis of Eq. (1). The major theoretic result of this paper is the existence and qualitative dynamics of a unique solution x in the case of exponential b and p .

The integral Eq. (1) also arises in more complex *parallel replacement* of many machines [3,4]. Equations of type (1) were first obtained in [5] for a parallel replacement equipment problem. It was shown that, under simplifying assumptions, this equation produced simple optimal replacement formulas stated in engineering economics textbooks. Although some economic analysis of the equation was provided, no analytical investigation was performed in [5]. Such integral equations have later appeared in the *vintage capital models* of mathematical economics, which describe economic growth under

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technological change [1,2,6,7,4,8–10]. As stressed in [2], “...dynamic general equilibrium models with vintage technology often collapse into a mixed delay differential equation system, which cannot be in general solved either mathematically or numerically”.

Mathematical investigation of (1) and similar integral equations was started in [3,11] where the solvability was proved in simple cases only, including Case A of Theorem 2 below. These early results were summarized in monograph [4]. The paper [12] developed a systematic investigation technique for equations of type (1) and applied it to a simpler equation arising in the *expense minimization* problem (rather than the *output maximization* as in the present paper). The qualitative picture is different in these alternative optimization problems. Optimization analysis of some more complex replacement problems [9,13,14] also involves dual integral equations of type (1). In particular, [9] investigates the *output maximization* problem in a model with a *two-factor production function* and generates a system of two equations of type (1) with an additional unknown variable. A hierarchical vintage capital replacement model of paper [14] takes into account network effects and involves the *expense minimization* with a two-factor production function. The paper [13] investigates an optimization model with a *three-factor production function* that depends on capital, labour, and energy consumption. The corresponding system of equations possesses two additional unknowns as compared with (1). The existence results are obtained in [9,13,14] only in cases when the optimal lifetime is constant.

The present paper explains the origin of Eq. (1), provides its rigorous analysis, and develops an efficient numeric algorithm for its solution. A new outcome is that, in general case, the solution of (1) exists only on the semi-infinite intervals (t_{cr}, ∞) or $(-\infty, t_{cr})$ depending on the comparative dynamics of the given b and p .

The obtained results are relevant for the theory of continuous vintage capital models. On the other hand, a number of replacement models in operations research [15–22] use the discrete time and lead to integer programming problems that are difficult to investigate analytically and numerically. The most recent contributions to replacement theory assume continuous time stochastic environment with no technological change [23,24] or use discrete models and mostly numeric approach to model the technological change [18,25]. Considering the replacement process in continuous time, we reduce corresponding optimization problems to integral equations for the optimal capital lifetime. Thus, the paper also demonstrates new perspectives that continuous analysis can bring into the analysis of discrete equipment replacement models.

The structure of the paper is as follows. The next section introduces the optimization problem for the single-machine replacement, derives its extremum conditions, and shows that the optimal replacement policy follows the solution of Eq. (1). In Section 3, the solvability of (1) is analyzed and qualitative properties of its solutions are established. An applied interpretation of obtained results is provided. Numerical algorithm and simulation example are presented in Section 4. The algorithm is tested on data about a typical US manufacturing plant and shows a good resemblance with engineering practice and literature sources. The simulation example confirms analytical results of the paper. The last section summarizes the results.

2. Integral equation for optimal replacement time

Let us formalize a mathematical model for the single-machine replacement problem mentioned in Introduction. Our goal is to find the optimal sequence $L = \{L_k, k = 1, 2, \dots\}$ of the lifetimes L_k of sequentially replaced machines. The sequence L can be either (a) an infinite series $\{L_k\}, k = 1, \dots, \infty$, of finite lifetimes or (2) N replacements $\{L_k\}, k = 1, \dots, N, N \geq 0$, with the infinite last lifetime $L_N = \infty$. Assuming that the first machine was purchased at the known time $\tau_0 \leq 0$, the replacement times are

$$\tau_{k+1} = \tau_k + L_{k+1}, \quad k = 0, 1, 2, \dots \quad (2)$$

Two common criteria of optimal machine replacement are the maximization of profit and the minimization of expenses [18]. Here we consider the problem of finding the optimal policy $\pi^* = \{L_k^*, k = 1, 2, \dots\}$, that maximizes the *discounted firm's net profit over the infinite horizon* $[\tau_0, \infty)$,

$$\text{maximize}_{L_j, j=1, \dots, \infty} J(\pi), \quad (3)$$

$$J(\pi) = \sum_{i=0}^{\infty} \int_{\tau_i}^{\tau_{i+1}} e^{-ru} b(\tau_i, u) du - \sum_{i=1}^{\infty} e^{-r\tau_i} p(\tau_i), \quad (4)$$

$\tau_{i+1} = \tau_i + L_{i+1}$. The first term of (4) is the discounted total product output and the second term is the discounted total cost of purchased machines (capital). The discount rate $r, r > 0$, is given. The efficiency $b(t, u)$ increases in t because of the TC (for a fixed machine age $u - t$), while physical deterioration decreases $b(t, u)$ when the machine age $u - t$ increases. Specific cases of $b(t, u)$ will be considered in Section 3.

Analogous machine replacement models in the *discrete time* [26–28,25, and others] are subjected to the additional restriction on the integer-valued unknowns L_k^* , which essentially complicates their analysis. Our continuous model (3) and (4) is not restricted to the integer-valued L_k^* and allows us to obtain deeper mathematical results.

2.1. Derivation of integral equation

Let us assume that the given functions $p(t)$ and $b(t, u)$ are continuously differentiable in t and continuous in u . Applying the standard optimization technique to problem (3) and (4), we obtain

Lemma 1. (the necessary condition for an extremum). If an optimal policy $\pi^* = \{L_k^*, k = 1, 2, \dots\}$ exists, then $L_k^* > 0, k = 1, 2, \dots$, and every finite component L_k^* satisfies the condition

$$\partial J / \partial \tau_i = 0, \quad k = 1, 2, \dots, \tag{5}$$

where

$$\partial J / \partial \tau_i = \int_{\tau_i}^{\tau_{i+1}} e^{-ru} \frac{\partial b(\tau_i, u)}{\partial \tau_i} du - e^{-r\tau_i} [p'(\tau_i) - rp(\tau_i) + b(\tau_i, \tau_i) - b(\tau_{i-1}, \tau_i)] \tag{6}$$

Proof. Rewriting (4) as

$$J = \int_{\tau_0}^{\tau_1} e^{-ru} b(\tau_0, u) du + \int_{\tau_1}^{\tau_2} e^{-ru} b(\tau_1, u) du + \dots + \int_{\tau_{i-1}}^{\tau_i} e^{-ru} b(\tau_{i-1}, u) du + \int_{\tau_i}^{\tau_{i+1}} e^{-ru} b(\tau_i, u) du + \dots - e^{-r\tau_1} p(\tau_1) - \dots - e^{-r\tau_i} p(\tau_i) - \dots,$$

and differentiating it,

$$\frac{\partial J}{\partial \tau_i} = e^{-r\tau_i} b(\tau_{i-1}, \tau_i) - e^{-r\tau_i} b(\tau_i, \tau_i) + \int_{\tau_i}^{\tau_{i+1}} e^{-ru} \frac{\partial b(\tau_i, u)}{\partial \tau_i} du + re^{-r\tau_i} p(\tau_i) - e^{-r\tau_i} p'(\tau_i),$$

we obtain (6). □

It is easy to see that $\partial J / \partial \tau_1 < 0$ and the optimal policy is trivial $L_1^* = \infty$ (no replacement) in the case with no TC that occurs at $b(t, u) = \text{const}$ and $p(t) = \text{const}$. The next statement derives the integral Eq. (1) for the optimal replacement times.

Theorem 1. Let $e^{-rt} p(t) \rightarrow 0$ and $\int_t^\infty e^{-ru} b(t, u) du \rightarrow 0$ at $t \rightarrow \infty$. If the optimal policy $\pi^* = \{L_k^*, k = 1, 2, \dots\}$ exists, then it is determined as

$$L_k^* = \tau_k - \tau_{k-1}, \tau_k = x^{-1}(\tau_{k-1}), \quad k = 1, 2, \dots, \tag{7}$$

where $x(t), t \in [\tau_0, \infty)$, is the solution of the nonlinear integral equation

$$\int_t^{x^{-1}(t)} e^{-ru} [b(t, u) - b(x(u), u)] du = e^{-rt} p(t), \quad t \in [\tau_0, \infty), \tag{8}$$

and function x^{-1} is the inverse of x .

Proof. Let us rewrite Eq. (8) as $F(t) = 0, t \in [\tau_0, \infty)$, where

$$F(t) = \int_t^{x^{-1}(t)} e^{-ru} [b(t, u) - b(x(u), u)] du - e^{-rt} p(t).$$

Differentiating it, we obtain

$$F'(t) = \int_t^{x^{-1}(t)} e^{-ru} \frac{\partial b(t, u)}{\partial t} du + e^{-rx^{-1}(t)} [b(t, x^{-1}(t)) - b(x(x^{-1}(t)), x^{-1}(t))] \frac{dx^{-1}(t)}{dt} - e^{-rt} [b(t, t) - b(x(t), t)] + re^{-rt} p(t) - e^{-rt} p'(t) = 0$$

The second term is zero because $x(x^{-1}(t)) = t$, hence,

$$F'(t) = \int_t^{x^{-1}(t)} e^{-ru} \frac{\partial b(t, u)}{\partial t} du - e^{-rt} [p'(t) - rp(t) + b(t, t) - b(x(t), t)] = 0, \tag{9}$$

$t \in [\tau_0, \infty)$. Since $F(t) \rightarrow 0$ at $t \rightarrow \infty$ under the theorem conditions, equality (9) is equivalent to the integral Eq. (8) (see Remark 1 below).

Now, let us compare equality (9) and conditions (5) and (6). Let $t = \tau_i$ in (9), then $x(\tau_i) = \tau_i - L(\tau_i) = \tau_{i-1}$ and $x^{-1}(\tau_i) = \tau_{i+1}$ by (2). Therefore, equality (9) coincides with conditions (5) and (6) at $t = \tau_k, k = 1, 2, \dots$. By Lemma 1, condition (5) holds true for the optimal policy at $t = \tau_k, k = 1, 2, \dots$. Hence, (9) also holds at $t = \tau_k, k = 1, 2, \dots$. The theorem is proven. □

Theorem 1 states that the optimal sequences $\{L_k^*\}$ and $\{\tau_k\}, k = 1, 2, \dots$, at given τ_0 are uniquely determined via the solution of (8). We will be interested in the continuously differentiable monotonic solutions $x(t)$ of Eq. (8), $t \in (-\infty, \infty)$, such that $x(t) < t$ and $x'(t) \geq 0$. Then the inverse $x^{-1}(t) > t$ because of the properties of the inverses (a function and its inverse are symmetric about the 45° line). Both $x(t)$ and $x^{-1}(t)$ are shown in Fig. 1. The vertical axis y is measured in the same time units as the horizontal axis t .

By (7), the machine installed at instant $t = \tau_{k-1}$ will be replaced at time $x^{-1}(t)$. Since t is any time in Eq. (8), let us consider the instant $t = x(u)$. Then, the machine installed at $x(u)$ will be replaced at time $x^{-1}(x(u)) = u$ (see Fig. 1). The function $t - x(t)$ is

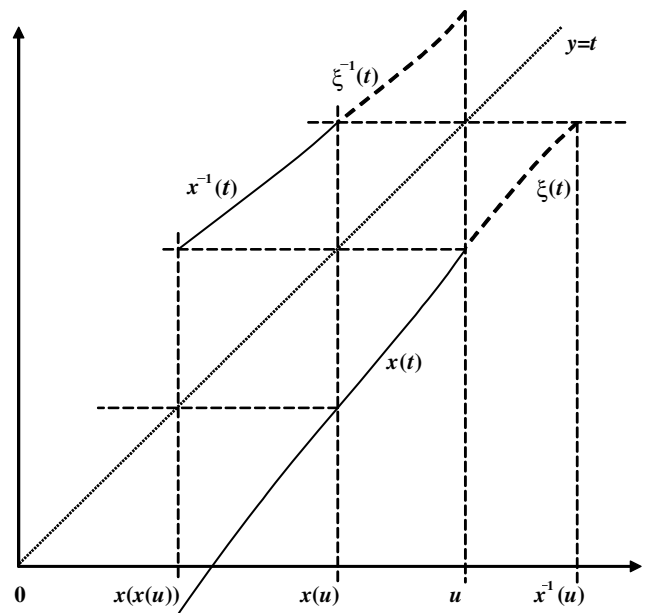


Fig. 1. The structure of the solution of Eq. (12) in the case B ($c_b > c_p$). The dashed lines are the initial function $\zeta(t)$ and its inverse $\zeta^{-1}(t)$, the solid lines are $x(t)$ and its inverse $x^{-1}(t)$, and the dotted straight 45° line highlights the symmetry between $x(t)$ and $x^{-1}(t)$.

the lifetime of the machine installed at instant $x(t)$ and replaced at time t (it is the vertical segment between the straight line $y = t$ and function $y = x(t)$ in Fig. 1). Correspondingly, $x^{-1}(t) - t$ is the ‘future’ lifetime of a ‘new’ machine installed at t and replaced at time $x^{-1}(t)$. Both lifetimes $t - x(t)$ and $x^{-1}(t) - t$ are unknown and can be expressed one through the other. The connection between $x(t)$ and $x^{-1}(t)$ is investigated in details in Section 3 (the proof of Theorem 2) and Section 4.1.1 (algorithms).

Remark 1. We investigate both Eqs. (8) and (9) depending which one is easier and clearer in every specific case. There are several historical, methodological, and computational reasons for analyzing Eqs. (8) and (9) together:

- In multi-machine parallel replacement problems, Eq. (8) arises *directly* as the result of applying necessary optimality conditions [3,4,11]. Since the parallel problems were investigated starting 1975 [5], integral equations of type (8) were the first to analyze. On the other hand, the original integral equations were differentiated in [3,5,12] to obtain simpler recurrent formulas for optimal trajectories.
- The integral Eq. (8) possesses a clear economic interpretation [4]. The left-hand side of (8) describes the difference between the discounted future total O&M costs of a new machine and the total O&M costs of the existing machine. Equality (8) states that, in the rational policy of machine replacement, this difference should be equal to the discounted price of the new machine.
- As demonstrated in Section 4, Eq. (9) is computationally simpler. So, it is profitable to start with the initial approximation of the solution to the integral Eq. (8) on some initial interval and after use its differentiated version (9) for continuing the trajectory to the entire interval (what is done in Section 4.1.1).

3. Theoretic analysis

By Theorem 1, Eq. (8) delivers the optimal replacement policy in the single-machine replacement problem (3) and (4). It also plays an essential role in multi-machine replacement [29,12] and some more general optimization problems of the economic growth theory [1–3,9,13].

In this section, we investigate the existence and dynamics of solutions to Eq. (8). We restrict ourselves with *the exponential TC* and exponential capital deterioration:

$$b(\tau, t) = b_0 e^{c_b \tau - c_d(t-\tau)}, \quad p(t) = p_0 e^{c_p t}, \tag{10}$$

$$b_0 > 0, \quad p_0 > 0, \quad c_b + c_d > 0, \quad c_b < r, \quad c_p < r, \quad b_0(r - c_b) > p_0(r + c_d)(r - c_p)e^{(c_p - c_b)t_0}. \tag{11}$$

Case (10) is important in applications and common in the replacement literature, see [26–28,25, and the references therein]. The coefficient c_b reflects the influence of TC on the productivity. Because of advances in science and technology, the newer

capital (equipment) is more efficient, although it can be more expensive. The coefficient c_p shows the rate of change in the price of new equipment. By (11), the efficiency and cost rates are smaller than the discount rate r . The coefficient c_d describes the impact of the age $t - \tau$ of capital on its efficiency (deterioration and learning-by-doing effects). The last inequality of (11) requires the initial efficiency b_0 to be superior in some sense to the initial equipment price p_0 (used in Case B of the below Theorem 2).

Then, Eq. (8) has the following form:

$$b_0 \int_t^{x^{-1}(t)} e^{-(r+c_d)(u-t)} [e^{(c_b+c_d)t} - e^{(c_b+c_d)x(u)}] du = p_0 e^{(c_p+c_d)t}, \quad t \in [t_0, \infty). \tag{12}$$

Theorem 2. Let conditions (10) and (11) hold. Then:

(A) If $c_b = c_p = c$, then Eq. (12) has a unique solution $x(t) = t - L$, $t \in (-\infty, \infty)$, where the constant $L > 0$ is determined from the following nonlinear equation:

$$(r + c_d)e^{-(c+c_d)L} - (c + c_d)e^{-(r+c_d)L} = (r - c) \left[1 - \frac{(r + c_d)p_0}{b_0} \right] \tag{13}$$

In particular, $L = \sqrt{2p_0/b_0/(c + c_d)} + o(r)$ at $r \ll 1$.

(B) If $c_b > c_p$, then Eq. (12) has a unique solution $x(t)$, $x(t) < t$, $x'(t) > 0$, $t \in [t_0, \infty)$, such that $L(t) = t - x(t)$ monotonically decreases and $x(t) \rightarrow t$ at $t \rightarrow \infty$.

(C) If $c_b < c_p$, then Eq. (12) has a unique solution $x(t)$, $x(t) < t$, on an interval $(-\infty, t_{cr})$, $t_{cr} > t_0$, such that $t - x(t)$ increases, $x'(t) \geq 0$ at $-\infty < t \leq t_d < t_{cr}$, $x^{-1}(t) = \infty$ and $x'(t) < 0$ at $t_d < t < t_{cr}$, $x(t) \rightarrow -\infty$ at $t \rightarrow t_{cr} - 0$ and $x(t) \rightarrow t$ at $t \rightarrow -\infty$. The critical time is $t_{cr} = \frac{1}{c_p - c_b} \ln \frac{b_0(r - c_b)}{p_0(r + c_d)(r - c_p)}$.

Remark 2. Theorem 2 covers the behavior of a solution to (8) on the interval $(-\infty, \infty)$. Although it is enough in practice to solve the equation on the interval $[t_0, \infty)$, it is interesting from a mathematical viewpoint to analyze the problem on a larger interval (when possible).

Remark 3. Case C involves a decreasing part of the trajectory $x(t)$ on the interval $[t_d, t_{cr})$, where the time t_d will be defined during the proof below. From the economic point of view, this part of trajectory has a little practical sense, because then there is no future replacement (since $x^{-1}(t) = \infty$) and the optimal strategy is bringing back older and older machines (since trajectory $x(t)$ is decreasing). We analyzed this part of trajectory mostly for mathematical completeness.

Proof. The investigation technique for another integral equation with simpler properties was developed in [10]. Here we extend this technique. For convenience, we transform Eq. (12) to the following form

$$b_0 \int_t^{x^{-1}(t)} e^{-\tilde{r}(u-t)} [e^{c_1 t} - e^{c_1 x(u)}] du = p_0 e^{c_2 t}, \quad t \in [t_0, \infty). \tag{14}$$

using the notations $c_1 = c_b + c_d$, $c_2 = c_p + c_d$, $\tilde{r} = r + c_d$.

The differentiation of (14) leads to

$$\tilde{r} e^{-c_1 L(t)} - c_1 e^{-\tilde{r} L(x^{-1}(t))} = \tilde{r} - c_1 - \tilde{r} \frac{p_0}{b_0} (\tilde{r} - c_2) e^{(c_2 - c_1)t} \tag{15}$$

with respect to the unknown function $L(t) = t - x(t) > 0$. So, if a solution of (14) exists, it satisfies (15). Depending on the relation between c_1 and c_2 , the solution has distinct dynamics and we investigate different cases separately.

Case A: $c_2 = c_1$. The existence of the unique solution $x(t) = t - L$, $L = \text{const}$, on the infinite interval $(-\infty, \infty)$ was proved in [3] and (13) is easily verified by direct substitution of $x(t)$ into (15).

Let us show that, at $c_2 \neq c_1$, (14) does have a solution on the entire infinite interval $(-\infty, \infty)$ but can have a solution on its infinite subintervals. It is easy to see that the left-hand side of (15) satisfies $-c_1 < \tilde{r} e^{-c_1 L(t)} - c_1 e^{-\tilde{r} L(x^{-1}(t))} < \tilde{r}$. Its right-hand side $\tilde{r} - c_1 - \tilde{r} \frac{p_0}{b_0} (\tilde{r} - c_2) e^{(c_2 - c_1)t} < -c_1$ at $\begin{cases} t \leq t_f & \text{if } c_1 > c_2 \\ t \geq t_f & \text{if } c_1 < c_2 \end{cases}$, where $t_f = \frac{1}{c_2 - c_1} \ln \frac{b_0}{p_0(\tilde{r} - c_2)}$. That is, Eq. (15) does not have a solution on $(-\infty, \infty)$ in these cases.

Case B: $c_1 > c_2$. Let us investigate the behavior of a solution to (15) on the interval $[t_0, \infty)$. Because of the last inequality of (11),

$$t_0 > \frac{1}{c_2 - c_1} \ln \frac{b_0(\tilde{r} - c_1)}{p_0 \tilde{r}(\tilde{r} - c_2)} > \frac{1}{c_2 - c_1} \ln \frac{b_0}{p_0(\tilde{r} - c_2)} = t_f$$

and the right-hand side of (15) is $0 < \tilde{r} - c_1 - \tilde{r} \frac{p_0}{b_0} (\tilde{r} - c_2) e^{(c_2 - c_1)t} < \tilde{r} - c_1$, $t \in [t_0, \infty)$. Hence, a positive solution $L(t)$ of (15) is possible. To prove the existence of the solution, let us provide the following steps:

Step 1. A recurrent formula for $x(t)$. The delay Eq. (15) connects $L(t) = t - x(t)$ and $L(x^{-1}(t)) = x^{-1}(t) - t$. It can be iteratively solved forward or backward. Let us consider its backward solution and rewrite (15) in the recurrent form

$$t - x(t) = -\frac{1}{c_1} \ln \left\{ 1 - \frac{p_0}{b_0} (\tilde{r} - c_2) e^{(c_2 - c_1)t} + \frac{c_1}{\tilde{r}} [e^{-\tilde{r}(x^{-1}(t) - t)} - 1] \right\}. \tag{16}$$

Step 2. A suitable initial function on a finite subinterval of $[t_0, \infty)$. As other delay equations, Eq. (14) can be solved with an initial condition $x(t) = \xi(t)$ given on an interval of a finite length. Let us consider a fixed instant $u > t_0$. To satisfy (14) at $t = u$, the initial function $x(t) = \xi(t)$, $\xi(t) < t$, should be given on an interval $[u, x^{-1}(u)]$, shown in Fig. 1. By Fig. 1, the interval length $x^{-1}(u) - u$ depends on the unknown function $x(t)$ itself. The continuously differentiable initial function $x(t) = \xi(t)$ can be constructed on $[u, \xi^1(u)]$ in a non-unique way. In the simplest case, the function $\xi(t)$ can be chosen as linear, then

$$\xi'(t) > 1, \quad t \in [u, x^{-1}(u)]. \tag{17}$$

A specific technique of constructing the initial function $x(t) = \xi(t)$, $t \in [u, \xi^1(u)]$, is illustrated in Section 4.1, where a numeric algorithm is constructed.

Step 3. A solution $x(t)$ on $[t_0, u)$. We will solve Eq. (15) from the right to the left. By $x(t) < t$, if we know $x(t) = \xi(t)$ at $t \in [u, x^{-1}(u)]$, we also know $\xi^{-1}(t)$ for $t \in [x(u), u]$ (see Fig. 1). Then, applying (16), we obtain a continuous solution $x(t)$ for $t \in [x(u), u]$. By (11), $x(t) < t$ at $t \in [x(u), u]$. The differentiation of (16) leads to

$$1 - x'(t) = \frac{e^{-\tilde{r}(x^{-1}(t) - t)} [1/x'(x^{-1}(t)) - 1] - \tilde{r}p_0/(c_1 b_0) (\tilde{r} - c_2) (c_2 - c_1) e^{(c_2 - c_1)t}}{\tilde{r} - \tilde{r}p_0/b_0(r - c_2) e^{(c_2 - c_1)t} + c_1 [e^{-\tilde{r}(x^{-1}(t) - t)} - 1]}. \tag{18}$$

The denominator of (18) is positive by (8) and its numerator is negative by (17). Hence, $x'(t) > 1$ at $t \in (x(u), u)$ and condition (17) is now satisfied for the function $x(t)$, $t \in [x(u), u]$. So, $x(t)$ can be taken as $\xi(t)$ on the new interval $(x(u), u)$ and the recurrent process (16) can be repeated to produce the solution $x(t)$ on $[x(x(u)), x(u)]$, then on $[x(x(x(u))), x(x(u))]$, and so on. The solution is illustrated in Fig. 1.

Step 4. The convergence of the iteration process (16). To prove the convergence, we give a small variation $\delta x(t)$ to the solution $x(t)$ of (15) and the corresponding $\delta\phi(t)$ to $x^{-1}(t)$. By the property of inverses, $x^{-1}(x(t)) = 1$ and $x^{-1}(x + \delta x) + \delta\phi(x + \delta x) = 1$. Applying the Taylor series to the last expression, we obtain $x^{-1}(x(t)) + \frac{dx^{-1}(x(t))}{dx} \delta x(t) + \delta\phi(x(t)) = 1 + o(\delta x, \delta\phi)$, from which, using the implicit function theorem, we find the following relation between $\delta\phi$ and δx :

$$\delta\phi(x(t)) \approx - \left[\frac{dx(t)}{dt} \right]^{-1} \delta x(t). \tag{19}$$

Giving $x + \delta x$ and the corresponding $x^{-1} + \delta\phi$ to (15) and subtracting (15) from the resulting expression, we have

$$\tilde{r} e^{-c_1(t-x(t)-\delta x(t))} - c_1 e^{-\tilde{r}(x^{-1}(t)+\delta\phi(t)-t)} - \tilde{r} e^{-c_1(t-x(t))} + c_1 e^{-\tilde{r}(x^{-1}(t)-t)} = 0.$$

Using (19) and the Taylor series approximations

$$\begin{aligned} \tilde{r} e^{-c_1(t-x(t))} (e^{-c_1 \delta x(t)} - 1) &= c_1 e^{-\tilde{r}(x^{-1}(t)-t)} (e^{-\tilde{r} \delta\phi(t)} - 1), \\ \tilde{r} c_1 e^{-c_1(t-x(t))} \delta x(t) &\approx c_1 \tilde{r} e^{-\tilde{r}(x^{-1}(t)-t)} \delta\phi(t), \end{aligned}$$

we finally obtain that

$$|\delta x(t)| \approx e^{c_1(x(t)-t)-\tilde{r}(x^{-1}(t)-t)} [x'(x^{-1}(t))]^{-1} |\delta x(x^{-1}(t))| \text{ for } |\delta x(t)| \ll 1. \tag{20}$$

Now, using $|x'| > 1$ from (17), we conclude that the iterations (16) converge, $|\delta x(t)| < |\delta x(x^{-1}(t))|$, when $c_1(x(t) - t) - \tilde{r}(x^{-1}(t) - t) < 0$, that is, when

$$L(t) < L(x^{-1}(t)) \tilde{r} / c_1. \tag{21}$$

It means that, when we solve (15) using iterations (16) from right to left, the small variation of x will decrease and the process converges if (21) holds. We justify (21) in the next step.

Step 5. The solution on the infinite interval $[t_0, \infty)$. The idea is to extend the solution obtained in Step 2 to an infinite interval. Let us first consider the infinite interval $[t_r, \infty)$, where $t_r \gg 1$. By (18), $L'(t) = 1 - x'(t) \rightarrow 0$ as $t \rightarrow \infty$, i.e., $L(t) \rightarrow \text{const}$ at $t \rightarrow \infty$, hence, the convergence condition (21) holds true on $[t_r, \infty)$. Therefore, the unique solution $x(t)$ on $[t_r, \infty)$ is obtained by the recurrent process (16) with the initial function ξ on $[u, x^{-1}(u)]$ by letting u tend to ∞ . Finally, the solution $x(t)$ on the remaining finite interval $[t_0, t_r)$ is uniquely determined from (16) as in Step 2.

Case C: $c_1 < c_2$. Although the dynamics of the solution in this case is different from the previous one, we use some steps of Case B. First of all, from (11) and (14)

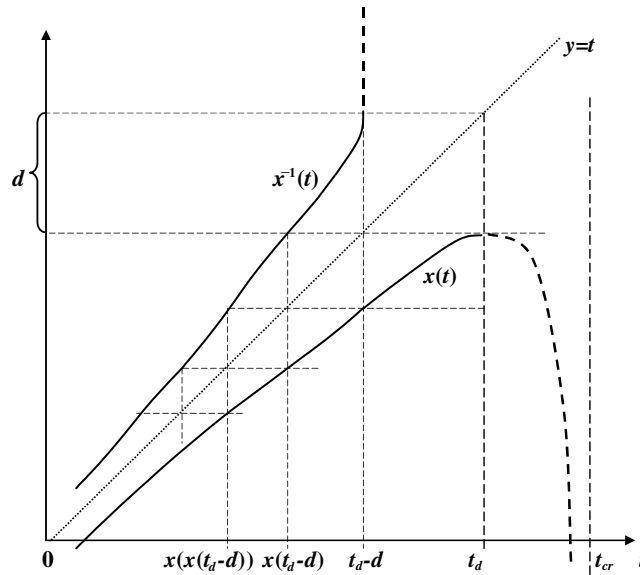


Fig. 2. The structure of the solution of Eq. (12) in case C ($c_b < c_p$).

$$t_0 < \frac{1}{c_2 - c_1} \ln \frac{b_0(\tilde{r} - c_1)}{p_0\tilde{r}(\tilde{r} - c_2)} < \frac{1}{c_2 - c_1} \ln \frac{b_0}{p_0(\tilde{r} - c_2)} = t_f.$$

By (16), a solution $x(t)$ satisfies $x(t) \leq t - d < t_{cr} - d$ at t close to t_{cr} , where the positive constant d will be defined below. Then, $x^{-1}(t)$ does not exist (see Fig. 2) and $x^{-1}(t) = \infty$ in the upper limit of (12) at $t_{cr} - d \leq t < t_{cr}$. Differentiating (12) in this case leads to

$$\tilde{r}e^{-c_1 L(t)} = \tilde{r} - c_1 - \tilde{r}p_0/b_0(\tilde{r} - c_2)e^{(c_2 - c_1)t}, \tag{22}$$

which produces the unique solution $L(t) = t - x(t)$ on $[t_{cr} - d, t_{cr}]$:

$$L(t) = -\frac{1}{c_1} \ln \left(1 - \frac{c_1}{\tilde{r}} - \frac{p_0}{b_0}(\tilde{r} - c_2)e^{(c_2 - c_1)t} \right) \tag{23}$$

It is easy to see that $L(t) \rightarrow \infty$ and $x(t) \rightarrow -\infty$ as $t \rightarrow t_{cr}$. Differentiating (22) gives

$$c_1(1 - x'(t))e^{-c_1 L(t)} = \frac{p_0}{b_0}(c_2 - c_1)(\tilde{r} - c_2)e^{(c_2 - c_1)t}. \tag{24}$$

The substitution of $e^{-c_1 L(t)}$ from (22) into (24) shows that $x'(t) = 0$ at $t = t_d < t_{cr}$, $x'(t) > 0$ at $t < t_d$, and $x'(t) < 0$ at $t_d < t < t_{cr}$, where the instant $t_d = t_{cr} - \frac{1}{c_2 - c_1} \ln \frac{c_1}{c_2}$ is obtained by substituting $x'(t) = 0$ and (14)–(24). Then, function (23) is the unique solution of Eq. (12) at $t \in [t_d - d, t_{cr}]$. The constant $d = -\frac{1}{c_1} \ln \left(1 - \frac{c_2(\tilde{r} + c_2 - c_1)}{c_1\tilde{r}} \right) > 0$ is found by substituting t_d in (23). Using (23) as the initial function $\xi(t) = x(t)$ at $t \in [t_d - d, t_{cr}]$ and the recurrent formula (16), we obtain a unique continuous solution $x(t)$ of (12) on $(-\infty, t_d - d]$ (see Fig. 2).

Let us analyze the behaviour of $x(t)$ at $t \rightarrow -\infty$. At $t \in [x(t_d), t_d]$, $x'(t) < 1$ and $L(t)$ increases, then, $x'(t) < 1$ holds true by (18) and $L(t)$ increases for all t . Hence, the convergence condition (21) holds true and the iterations (16) converge for all t . By (18), $L'(t) = 1 - x'(t) \rightarrow 0$ and $L(t) \rightarrow 0$ at $t \rightarrow -\infty$. The theorem is proven. \square

The constructed solution of (12) in Cases B and C is illustrated in Figs. 1 and 2.

The integral Eq. (8) is a key for the optimal replacement decision. Its properties allow us to analyze the dynamics of the optimal machine lifetime $L(t) = t - x(t)$, in particular, its dependence on the intensity of TC and discounting. Theorem 2 produces simple qualitative properties of optimal replacement policies. Specifically, the following outcomes hold.

- *Optimal variable lifetime.* If the efficiency $b(t, u)$ of machines increases in t faster than its price $p(t)$, i.e., $c_b < c_p$, then the optimal machine lifetime $L(t)$ decreases.
- If $p(t)$ increases faster than $b(t, u)$, $c_b > c_p$, then the optimal machine lifetime $L(t)$ increases indefinitely and ceases to exist after some instant t_{cr} .
- *Optimal constant lifetime.* At equal TC rates $c_b = c_p$, the optimal lifetime is constant and determined by Eq. (13).
- *TC impact on the optimal lifetime.* If $c_b = c_p = c$, then the optimal machine lifetime L^* is smaller in the case of more intensive TC (a larger rate c).

4. Numeric simulation

At our knowledge, there are no results on the numeric solution of integral equations of type (8) in literature. This paper is the first attempt to formalize corresponding algorithms. We construct algorithms for solving Eq. (8) on finite and infinite intervals. The first problem is an initial problem for the delay Eq. (8) with a given “initial function”. The second problem estimates the unique solution on the infinite interval $[t_0, \infty)$ (if it exists, see Theorem 1) without an initial function.

4.1. Approximate algorithms

We will consider numeric solution of Eq. (8) in a more general situation than the exponential case (10). The only necessary assumption is the strict monotonicity of $b(t, u)$ in t (i.e., the presence of TC). The functions $p(t)$ and $b(t, u)$ are assumed to be continuously differentiable. We construct algorithms for solving the initial problem for Eq. (8) (Problem A) and for finding a unique infinite interval solution (Problem B). At the same time, Problem A is an auxiliary step for Problem B.

4.1.1. The initial problem A

The problem consists of finding a solution $x(t)$ of the delay Eq. (8) on $[t_1, t_0)$, $t_1 < t_0$, or $[t_0, t_1)$, $t_1 > t_0$, at the given initial monotonic function $x(t) = \xi(t) < t$ at $t \in [t_0, x^{-1}(t_0)]$. A general numeric approach to initial problems for delay equations is the *method of steps*. It allows computing exact solutions of constant delay equations and has been recently developed for solving the equations with variable delay and the functional delay differential equations with state dependent delays (it is known as the “standard approach” in [30,31]). Comparing to the case of constant delay equations, the novelty of Eq. (8) is that the length of the initial interval $[t_0, \xi^{-1}(t_0)]$ itself depends on the initial function.

Applying the method of steps to problem A is straightforward. The algorithm is based on the recurrent formula

$$b(t, t) - b(x(t), t) = \int_t^{x^{-1}(t)} e^{-r(u-t)} \frac{\partial b(t, u)}{\partial t} du + rp(t) - p'(t), \quad (25)$$

obtained from (6). The only nonstandard step is to invert the implicit function $b(x, t)$ in x for each fixed t (for backward solution) or $\int_t^x f(t, u) du$ (for forward solution), which can be done using a polynomial interpolation.

For definiteness, let us assume that $t_1 < t_0$ (backward solution) and the initial function $x(t) = \xi(t)$, $t \in [t_0, x^{-1}(t_0)]$, satisfies (8) at $t = t_0$. Then, by the definition of the inverse, the function $x^{-1}(t)$ is known at $t \in [x(t_0), t_0]$ and the inversion of $b(x, t)$ in formula (25) produces the solution $x(t) < t$ of (8) on the first-step interval $[x(t_0), t_0]$. In a similar manner, we obtain $x(t)$ on the second-step interval $[x(x(t_0)), x(t_0)]$, and so on. If $x(t)$ becomes non-monotonic at some point t' , then the inverse $x^{-1}(t_0)$ does not exist at the point $t = x(t')$ of the next step interval and the algorithm stops after finding the solution $x(t)$ on $[x(t'), t_0]$.

The dynamics of the constructed solution $x(t)$ heavily depends on the given functions ξ , p , and q . In the general case, $x(t)$ has discontinuities at the instants $x^k(t_0)$, $k = 0, \pm 1, \pm 2, \pm 3, \dots$ (unless the function ξ is specially adjusted). Also, $x(t)$ may have discontinuities at instants, where the derivatives of p and b in (25) possess jumps. As shown in [31], the presence of discontinuity points leads to serious challenges in solving state-dependent delay equations, which are out of our scope. Instead, we focus on solving Eq. (8) on an infinite interval. By Theorem 1, the Eq. (8) has the unique solution \bar{x} on an infinite interval, at least, for exponential b and p . Therefore, the above constructed solution $x(t)$ of Problem A may not exist in a general case (unless the initial function $\xi(t)$ is a part of the unique solution \bar{x}). The problem of finding the unique solution \bar{x} is important for applications and is considered below.

4.1.2. The asymptotic problem B

consists of finding (or, at least, estimating) the solution $x(t)$ of Eq. (8) on the infinite interval $[t_0, \infty)$. In practice, this problem needs to be solved on a finite interval $[t_0, t_1]$ in the case when the given functions p and b are known on $[t_0, t_1]$ only (see an example in Section 4.2). Then, the additional assumption is made that the solution dynamics remains similar at $t > t_1$.

The algorithm for the asymptotic problem B includes two steps:

Step 1. Constructing a proper initial function $x(t) = \xi(t)$, $t \in [x(t_1), t_1]$.

Step 2. Applying the recurrent formula (25) to find $x(t)$ on $[t_0, x(t_1)]$.

During Step 1, an initial function $x(t) = \xi(t)$ needs to be constructed on $[x(t_1), t_1]$, such that (8) is satisfied and $x(t)$ is continuous at point $\hat{t} = x(t_1)$. One can see that it can be done in a non-unique manner. We implement this step by choosing the linear

$$\xi(t) = \alpha t - L + (1 - \alpha)t_1 \text{ on } [\xi(t_1), t_1], \quad (26)$$

that depends on two parameters α and $L = t_1 - \xi(t_1)$. If $\frac{\partial q(t_1, t_1)}{\partial t} > p'(t_1)$, we select an initial value $\alpha > 1$, otherwise $0 < \alpha < 1$. Substituting ξ as x into (8), we obtain the equation $\sigma(\alpha, L) = 0$,

$$\Phi(\alpha, L) = \int_{t_1-L}^{t_1} e^{-ru} [b(t_1 - L, u) - b(t_1 - L - \alpha(t_1 - u), u)] du - e^{-r(t_1-L)} p(t_1 - L). \quad (27)$$

At the given α , we find the unique $L > 0$ and the point $\hat{t} = \xi(t_1) = t_1 - L$ from (27). The uniqueness of L , follows from the fact that $\sigma(\alpha, 0) < 0$ and $\partial\sigma/\partial L > 0$.

Next, we find $x(t)$ at $t \leq \hat{t}$ close to \hat{t} from the recurrent formula (25). In a general case, $x(\hat{t})$ differs from $\xi(\hat{t})$. If $x(\hat{t}) < \xi(\hat{t})$, then we slightly decrease α in (26), which increases L obtained from (27) and decreases $\xi(\hat{t})$. Otherwise, we decrease α . Applying any numeric technique for solving a single nonlinear equation (e.g., bisection algorithm) and repeating the process of finding L and \hat{t} and calculating $x(\hat{t})$ from (25), we get $x(\hat{t}) \approx \xi(\hat{t})$. The corresponding values α and L deliver the initial function (26) continuous at $\hat{t} = \xi(t_1)$. It completes Step 1.

Now we can implement Step 2, using the above constructed algorithm for Problem A. Iterating the recurrent formula (25) backward, we obtain a continuous function $x(t)$ for $[t_0, x(t_1)]$. When $t_1 - t_0$ becomes larger ($t_1 - t_0 \rightarrow \infty$), the obtained $x(t)$ tends to the unique solution $\bar{x}(t)$ of (11) on $[t_0, \infty)$ (if it exists at the given p and b). This fact is established by the same arguments as in the proof of Theorem 2.

The constructed algorithms are implemented in Visual Basic/Excel and can be provided to all interested readers.

4.2. Numeric example

To approbate our algorithm on real economic data, let us consider modeling of the optimal equipment lifetime for a typical US manufacturing plant. Then, the hypothesis about exponential technological change is commonly accepted [15,16,18,28,25] and we can use the assumptions (10). Under (10), the identification of Eq. (12) involves five parameters: the interest rater, the TC rates c_b and c_p , deterioration rate c_d , and the initial productivity/cost ratio b_0/p_0 . Following [1,6], we choose $r = 0.1$ and $c_d = 0$. For the rest of the parameters, the identification is challenging. Various sources provide the basic rate of technological change c_b in the range of 3–17% for the years 1970–2000. We follow [32] and choose $c_b = 0.08$. The rate c_p is even more difficult to find in literature and we provide a series of numeric experiments with different c_b . Finally, under given r, c_b, c_p , and c_d , the identification parameter b_0/p_0 is adjusted to produce the current equipment lifetime $t_1 - x(t_1)$ of approximately 24 years at the initial modeling year $t_0 = 0$.

Next, we apply the algorithm of Section 4.1.2 for finding the unique infinite-horizon solution \bar{x} at $t_0 < t < T$. The basic simulation run is shown in Fig. 3. The calculation parameters involve the horizon length $T = 100$ years and the discretization step $H = 0.1$. At the accepted parameter values, the algorithm does not require smaller steps and longer horizons [33]. To analyze the robustness of the algorithm, we have found the solution \bar{x} of Eq. (12) for different algorithm parameters and values of T . The results appear to be too close to show graphically. Some simulation results for different H of T are provided in the Table 1. One can see that the 10% change of the horizon length T has no impact on the calculated value of the optimal

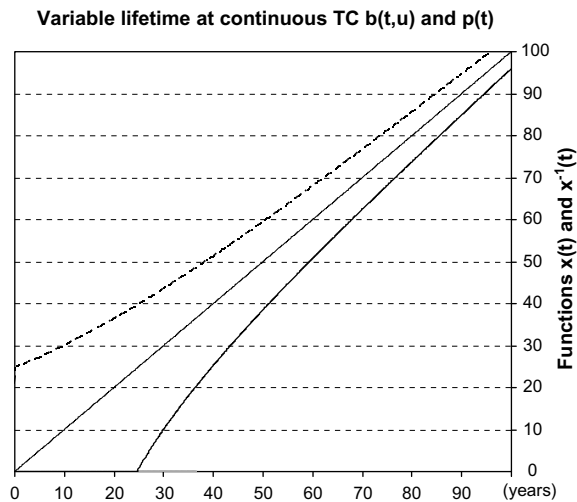


Fig. 3. Numeric solution of Eq. (12) at $r = 0.1, c_b = 0.08$, and $c_p = 0.05$. The solid line is x , the dashed line is the inverse x^{-1} of x , and the straight line highlights the symmetry between x and x^{-1} .

Table 1
Calculated results for different simulation periods and discretization steps

Horizon length $T - t_0$	Step H	Calculated $L(t_0)$
100	0.2	24.036
100	0.1	24.104
90	0.1	24.102
50	0.1	23.523

Table 2
Simulation results for different input data

Rate c_b	Rate c_p	Ratio b_0/p_0	Calculated α	Calculated $L(t_0)$	Calculated $L(T)$
0.08	0	0.5	1.11	22.3	0.4
0.08	0.01	0.5	1.03	24.10	0.69
0.08	0.05	1.1	1.07	22.83	3.87
0.08	0.08	1.65	1	23.10	23.02
0.08	0.085	1.7	0.76	24.34	39.55
0.08	0.087	1.78	0.62	23.14	46.86

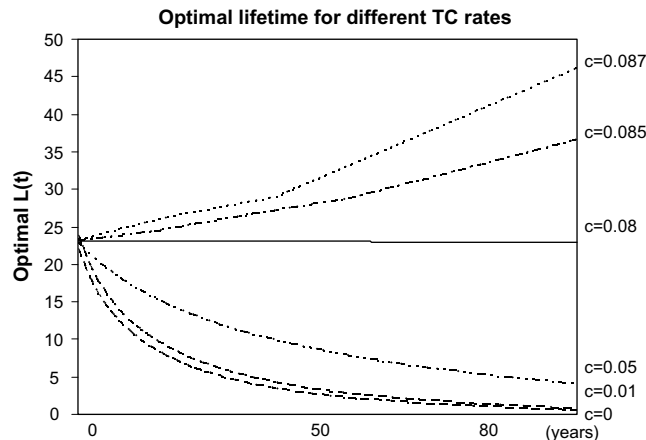


Fig. 4. The optimal machine lifetime $L(t) = t - x(t)$ for the efficiency rate $c_b = 0.08$ and different values of the capital cost rate $c = c_p$. The constant lifetime (solid line) corresponds to $c_p = 0.08$.

$L(t_0)$ at the initial modeling year $t_0 = 0$, and even cutting T twice still causes only 0.3% relative change in $L(t_0)$. When we increase T , the value of $L(t_0)$ simply stays the same. So, as predicted theoretically in [Theorem 2](#), the algorithm indeed delivers the unique solution \bar{x} (when the horizon length $T - t_0$ is at least 4–5 times larger than $\max L(t)$).

As follows from [Theorem 1](#), the ratio c_p/c_b impacts the core dynamics of the optimal replacement. If the rates of TC in the operating costs and machine price are different, $c_b = c_p$ in (12), then, by [Theorem 2](#), the optimal machine lifetime decreases at $c_b > c_p$ and increases at $c_b < c_p$. The dynamics of the optimal replacement was analyzed for several scenarios $c_b > c_p$, $c_b = c_p$, and $c_b < c_p$. The key input and output data are provided in the [Table 2](#). [Fig. 4](#) shows the variable optimal lifetime for six different scenarios. The simulation results confirm the analytic conclusions of [Theorem 2](#). In particular, at $c_b = c_p$, the optimal lifetime appears to be constant (then it can be also found from [Eq. \(13\)](#)). Naturally, the sensitivity of solution \bar{x} to the difference $c_b - c_p$ is higher when $c_b > c_p$ (and \bar{x} increases).

5. Conclusion

The paper develops new analytic and numeric techniques for finding optimal replacement policies. The techniques are based on deriving and solving a nonlinear integral equation for the optimal lifetime of machines. The obtained theoretic results lead to new insights into the qualitative dynamics of equipment replacement processes in manufacturing systems under the presence of technological change. In particular, the solution x of (1) exists on the infinite interval $(-\infty, \infty)$ only when the growth rates of the given $b(t, u)$ and $p(t)$ in t are the same (then $x(t) = t - \text{const}$). In more general cases, the solution x exists only on the semi-infinite intervals (t_{cr}, ∞) or $(-\infty, t_{\text{cr}})$ depending on the comparative dynamics of b and p ([Figs. 1 and 2](#)). In the alternative expense minimization problem [12], the optimal dynamics is qualitatively different and the solution exists on the whole interval $(-\infty, \infty)$. The numeric experiments on real industry data confirm the theoretical findings of the paper and the robustness of the developed algorithm.

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