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Age-structured Population Model with Cannibalism

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Abstract

An age-structured population model with cannibalism is investigated. We determine the steady states and study the local asymptotic stability as well as the global stability. The results in this paper generalize previous results.

Keywords: Population; Cannibalism; Stability; Steady state; Age-structure.

MSC 2000: 45K05; 45M05; 45M10; 45M20; 35B35; 35B40; 35L60; 92D25.

1. Introduction

In this paper, we study an age-structured population model with cannibalism that has been studied in Bekkal-Brikci, et al. (2007). The model is as follows

$$\begin{aligned}&\frac{\partial p(a,t)}{\partial a}+\frac{\partial p(a,t)}{\partial t}+\mu(a)p(a,t)=0, \quad a>0, \quad t>0,\\&p(0,t)=\varphi(P(t))B(t),\ t\geq 0,\\&p(a,0)=p_0(a), \quad a\geq 0,\end{aligned} \tag{1}$$

where $p(a,t)$ is the density of the population with respect to the chronological age $a\in[0,\infty)$ at time t, $P(t)=\int_0^\infty p(a,t)da$ is the total population size at time t, $\beta(a),\mu(a)$ are, respectively, the birth rate and the mortality rate, $B(t)=\int_0^\infty \beta(a)p(a,t)da$ is the number of births per unit time, and $\varphi(P(t))$ is a function that determines the effect of cannibalism on offspring at the population size $P(t)$.

We study problem (1) under the following general assumptions: $\beta(a)$ and $\mu(a)$ are nonnegative functions of $a \in [0, \infty)$, $\beta \in L_\infty[0, \infty)$, $\mu \in L^1_{loc}([0, \infty))$,

$$\int_0^\infty \mu(a)da = +\infty, \quad \int_0^\infty \pi(a)da < +\infty, \quad \text{where} \quad \pi(a) = e^{-\int_0^a \mu(\tau)d\tau}, \tag{2}$$

$$\varphi(t) > 0, \quad \varphi'(t) < 0, \forall t \geq 0, \quad \text{and} \quad \varphi(t) \longrightarrow 0 \text{ as } t \longrightarrow +\infty. \tag{3}$$

In Bekkal-Brikci, et al. (2007) problem (1) is described and existence and uniqueness results via degree theory are obtained as well as the local asymptotic stability of the nontrivial steady state and the global stability of the solution of the age-independent model via a delay differential equation when $\varphi(t) = e^{-\alpha t}, \alpha \geq 0$.

However, a more general model than (1) is considered in Rorries (1976), and Iannelli (1995), and local stability results as well as global stability results are obtained. The stability results given therein are for the age-dependent version of the model. Also, in Iannelli (1995) and Gurtin, et al. (1974), existence and uniqueness results via a fixed point theorem are obtained.

We note that solutions of problem (1) depend continuously on the initial age-distributions and this fact can be proved via the same method as in Di Blasio, et al. (1982). Accordingly, problem (1) is well posed.

In this paper, we study problem (1) and determine its steady states and examine their local and global stability. We show that if $\varphi(0) \leq R_0$, see section 2 for the definition of R_0, then the trivial steady state is the only steady state, and we show that it is globally stable if $\varphi(0) < R_0$. We also show that if $\varphi(0) > R_0$, then a trivial steady state as well as a nontrivial steady state are possible steady states, and the nontrivial steady state is unique when it exists. In this case the trivial steady state is unstable, and we give four sufficient conditions for the local asymptotic stability of the nontrivial steady state. We also give a sufficient condition for the global stability of the nontrivial steady state which generalizes the results given in Bekkal-Brikci, et al. (2007), Rorres (1976) and Iannelli (1995).

The organization of this paper as follows: in Section 2 we determine the steady states; in Section 3 we study the local stability of the steady states; in Section 4 we study the global stability of the steady states; in Section 5 we generalize our model; in Section 6 we conclude our results.

2. The Steady States

In this section, we determine the steady states of problem (1). A steady state of problem (1) satisfies the following:

$$\begin{cases} \dfrac{dp_\infty(a)}{da} + \mu(a)p_\infty(a) = 0, \quad a > 0, \\ p_\infty(0) = \varphi(P_\infty)B_\infty. \end{cases} \tag{4}$$

From (4), by solving the differential equation, we obtain that

$$p_\infty(a) = p_\infty(0)\pi(a), \tag{5}$$

where $\pi(a)$ is given by (2).

Also, from (4) and (5), we obtain that B_∞ satisfies the following:

$$B_\infty = \varphi(P_\infty)B_\infty \int_0^\infty \beta(a)\pi(a)da. \tag{6}$$

Accordingly, from (6), we conclude that either $B_\infty = 0$ or P_∞ satisfies the following:

$$1 = \varphi(P_\infty) \int_0^\infty \beta(a)\pi(a)da. \tag{7}$$

In the following theorem, we describe the steady states of problem (1). And in order to facilitate our writing, we define a threshold parameter R_0 by

$$R_0 = \frac{1}{\displaystyle\int_0^\infty \beta(a)\pi(a)da}. \tag{8}$$

Theorem 2.1:

(a) Suppose that

$$\varphi(0) \leq R_0, \tag{9}$$

then the trivial steady state $p_\infty(a) \equiv 0$ is the only steady state.

(b) Suppose that

$$\varphi(0) > R_0, \tag{10}$$

then the trivial steady state as well as the nontrivial steady state are possible steady states. The nontrivial steady state is unique when it exists.

Proof: To prove (a), we note that if $B_\infty = 0$, then by (4), we obtain $p_\infty(0) = 0$, and hence $p_\infty(a) \equiv 0$, by equation (5). Now, if $B_\infty \neq 0$, then the right-hand side of (7) is a decreasing function of P_∞, and so by (9), either $\dfrac{\varphi(0)}{R_0} < 1$, and accordingly, by (6), $B_\infty = 0$, or $\dfrac{\varphi(0)}{R_0} = 1$, and hence $P_\infty = 0$, and therefore $p_\infty(a) \equiv 0$ is the only solution of equation (7) since φ is a strictly decreasing function. This completes the proof of (a).

To prove (b), we note that if $B_\infty = 0$, then as is shown in (a), the only steady state is the trivial steady state $p_\infty(a) \equiv 0$. We also note that if $B_\infty \neq 0$, then the right-hand side of of equation (7) is a decreasing function of P_∞ which has a value greater than one if $P_\infty = 0$, by (10), and approaches zero as $P_\infty \longrightarrow +\infty$. Accordingly $\exists\ P_\infty > 0$ which satisfies (7) and which is necessarily unique, and hence it gives rise to a unique nontrivial steady state. This completes the proof of (b), and therefore the proof of the theorem is completed.

3. Local Stability Results

In this section, we study the local stability of the steady states for problem (1) as given by Theorem 2.1.

To study the stability of a steady state $p_\infty(a)$, which is a solution of (4) and is given by equation (5), we linearize problem (1) at $p_\infty(a)$ in order to obtain a characteristic equation, which in turn will determine conditions for the stability. To that end, we consider a perturbation $\omega(a,t)$ defined by $\omega(a,t) = p(a,t) - p_\infty(a)$, where $p(a,t)$ is a solution of problem (1). Accordingly, we obtain that $\omega(a,t)$ satisfies the following:

$$\begin{cases} \frac{\partial \omega(a,t)}{\partial a} + \frac{\partial \omega(a,t)}{\partial t} + \mu(a)\omega(a,t) = 0, \quad a > 0, \quad t > 0, \\ \omega(0,t) = \varphi(P_\infty)\int_0^\infty \beta(a)\omega(a,t)da + B_\infty \varphi'(P_\infty)\int_0^\infty \omega(a,t)da, \quad t \geq 0, \\ \omega(a,0) = p_0(a) - p_\infty(a), \quad a \geq 0. \end{cases} \tag{11}$$

By substituting $\omega(a,t) = f(a)e^{\lambda t}$ in (11), where λ is a complex number, and straightforward calculations, we obtain the following characteristic equation:

$$1 = \varphi(P_\infty)\int_0^\infty \beta(a)\pi(a)e^{-\lambda a}da + B_\infty \varphi'(P_\infty)\int_0^\infty \pi(a)e^{-\lambda a}da.() \tag{12}$$

In the following theorem, we describe the stability of the trivial steady state $p_\infty(a) \equiv 0$.

Theorem 3.1: The trivial steady state $p_\infty(a) \equiv 0$, is locally asymptotically stable if $\varphi(0) < R_0$, and is unstable if $\varphi(0) > R_0$.

Proof: We note that for the trivial steady state $p_\infty(a) \equiv 0$, $B_\infty = 0$, and therefore, from the characteristic equation (12) , we obtain the following characteristic equation:

$$1 = \varphi(0)\int_0^\infty \beta(a)\pi(a)e^{-\lambda a}da. \tag{13}$$

To prove the local asymptotic stability, we note that if $\varphi(0) < R_0$, then equation () can not be satisfied for any λ with $Re\lambda \geq 0$ since

$$\begin{aligned} \left|\varphi(0)\int_0^\infty \beta(a)\pi(a)e^{-\lambda a}da\right| &\leq \varphi(0)\int_0^\infty \beta(a)\pi(a)e^{-aRe\lambda}da \\ &\leq \frac{\varphi(0)}{R_0} < 1. \end{aligned}$$

Accordingly, the trivial steady state is locally asymptotically stable if $\varphi(0) < R_0$.

To prove the instability of the trivial steady state when $\varphi(0) > R_0$, we note that if we define a function $g(\lambda)$ by

$$g(\lambda) = \varphi(0)\int_0^\infty \beta(a)\pi(a)e^{-\lambda a}da,$$

and suppose that λ is real, then we can easily see that $g(\lambda)$ is a decreasing function if $\lambda > 0$, $g(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow +\infty$, and $g(0) = \dfrac{\varphi(0)}{R_0} > 1$. Therefore, if $\varphi(0) > R_0$, then there exists $\lambda^* > 0$ such that $g(\lambda^*) = 1$, and hence the trivial steady state $p_\infty(a) \equiv 0$ is unstable. This completes the proof of the theorem.

In the next theorem, we give our first sufficient condition for the local asymptotic stability of the nontrivial steady state. We note that this condition has been given in Iannelli (1995), where Rouché's Theorem is used, also in Bekkal-Brikci, et al. (2007) the same condition is given, where the Implicit Function Theorem is used. In our proof, we use some simple arguments which make the proof shorter and straightforward.

Theorem 3.2: Suppose that $\varphi(0) > R_0$ and $\mu^2(a) > \mu'(a)$, then the nontrivial steady state is locally asymptotically stable.

Proof: We note that if we let $\lambda = x + iy$ in the characteristic equation (12) , then we obtain the following equivalent pair of equations:

$$0 = \int_0^\infty e^{-xa}\pi(a)\sin ya[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]da, \tag{14}$$

$$1 = \int_0^\infty e^{-xa}\pi(a)\cos ya[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]da. \tag{15}$$

Now, suppose that $x \geq 0$ and $y = 0$, then from equations (7) and (15), we obtain

$$\begin{aligned} 1 &= \varphi(P_\infty)\int_0^\infty e^{-xa}\beta(a)\pi(a)da + B_\infty\varphi'(P_\infty)\int_0^\infty e^{-xa}\pi(a)da \\ &\leq \varphi(P_\infty)\int_0^\infty \beta(a)\pi(a)da + B_\infty\varphi'(P_\infty)\int_0^\infty e^{-xa}\pi(a)da \\ &= 1 + B_\infty\varphi'(P_\infty)\int_0^\infty e^{-xa}\pi(a)da \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (12) is not satisfied for any $x \geq 0$ and $y = 0$.

Now, we suppose that $x \geq 0$ and $y \neq 0$, and observe that equation (15) can be rewritten in the following form:

$$\begin{aligned} 1 &= \varphi(P_\infty)\int_0^\infty e^{-xa}\beta(a)\pi(a)\cos ya da + \\ &\quad \frac{B_\infty\varphi'(P_\infty)}{y^2}\int_0^\infty e^{-xa}\pi(a)[(x+\mu(a))^2 - \mu'(a)](1-\cos ya)da \\ &\leq 1 + \frac{B_\infty\varphi'(P_\infty)}{y^2}\int_0^\infty e^{-xa}\pi(a)[(x+\mu(a))^2 - \mu'(a)](1-\cos ya)da \\ &< 1. \end{aligned}$$

Note that the last inequality is obtained by using the assumption $\mu^2(a) > \mu'(a)$. Accordingly, the nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the following theorem, we give our second sufficient condition for the local asymptotic stability of the nontrivial steady state.

Theorem 3.3: Suppose that $\varphi(0) > R_0$ and

$$\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty) \geq 0, \forall a \in [0, \infty), \tag{16}$$

then the nontrivial steady state is locally asymptotically stable.

Proof: We note that using the characteristic equation (12) and assuming that $Re\lambda \geq 0$, we obtain

$$\begin{aligned} 1 &= \Big|\int_0^\infty e^{-\lambda a}\pi(a)[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]da\Big| \\ &\leq \int_0^\infty e^{-aRe\lambda}\pi(a)[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]da \\ &\leq 1 + B_\infty\varphi'(P_\infty)\int_0^\infty \pi(a)da \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (12) is not satisfied for any λ with $Re\lambda \geq 0$ if (16) is satisfied. Therefore, the nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the following theorem, we give our third sufficient condition for the local asymptotic stability of the nontrivial steady state.

Theorem 3.4: Suppose that $\varphi(0) > R_0$ and

$$\int_0^\infty \pi(a)\Big|\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)\Big|da < 1, \tag{17}$$

then the nontrivial steady state is locally asymptotically stable.

Proof: We note that using the characteristic equation (12) and assuming that $Re\lambda \geq 0$, we obtain

$$\begin{aligned} 1 &= \Big|\int_0^\infty e^{-\lambda a}\pi(a)[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]da\Big| \\ &\leq \int_0^\infty e^{-aRe\lambda}\pi(a)\Big|[\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)]\Big|da \\ &\leq \int_0^\infty \pi(a)\Big|\varphi(P_\infty)\beta(a) + B_\infty\varphi'(P_\infty)\Big|da \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (12) is not satisfied for any λ with $Re\lambda \geq 0$ if (17) is satisfied. Therefore, the nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the following theorem we give our fourth sufficient condition for the local asymptotic stability of the nontrivial steady state.

Theorem 3.5: Suppose that $\varphi(0) > R_0$ and $\beta(a) = \beta = constant$, then the nontrivial steady state is locally asymptotically stable.

Proof: In this proof, we use the characteristic equation (12) , in the form of the pair of equations (14) and (15), and note that if $\varphi(P_\infty)\beta + B_\infty\varphi'(P_\infty) = 0$, then equation (15) is not satisfied for any x and y, because the right-hand side of (15) is zero whereas the left-hand side is one.

Now, we suppose that $\varphi(P_\infty)\beta + B_\infty\varphi'(P_\infty) \neq 0$, and $x \geq 0$. Observe that equation (14) is not satisfied for any $y \neq 0$ since $\int_0^\infty e^{-xa}\pi(a)\sin yada \neq 0$, because $e^{-xa}\pi(a)$ is a decreasing function of a. And if we suppose that $y = 0, \varphi(P_\infty)\beta + B_\infty\varphi'(P_\infty) \neq 0$, and $x \geq 0$, then by using (15), we obtain

$$\begin{aligned} 1 &= \varphi(P_\infty)\beta\int_0^\infty e^{-xa}\pi(a)da + B_\infty\varphi'(P_\infty)\int_0^\infty e^{-xa}\pi(a)da \\ &\leq 1 + B_\infty\varphi'(P_\infty)\int_0^\infty e^{-xa}\pi(a)da \\ &< 1. \end{aligned}$$

Accordingly, the characteristic equation (12) is not satisfied for any λ with $Re\lambda \geq 0$ if β is a constant independent of age. Therefore, the nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

4. Global Stability Results

In this section, we study the global behavior of problem (1). In particular, we show that the trivial steady state, shown to be locally asymptotically stable in Theorem 3.1, is actually globally stable. Furthermore, we also show that the nontrivial steady state, shown to be locally asymptotically stable in Theorem 3.5, is also actually globally stable.

We start by noting that by integrating problem (1) along characteristic lines $t - a =$ constant, we obtain that $p(a,t)$ satisfies

$$p(a,t) = \begin{cases} p_0(a-t)\dfrac{\pi(a)}{\pi(a-t)}, & a > t, \\ \varphi(P(t-a))B(t-a)\pi(a), & a < t. \end{cases} \tag{18}$$

And by using equation (18), we obtain

$$P(t) = \int_0^t \pi(a)\varphi(P(t-a))B(t-a)da + \int_t^\infty p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da, \tag{19}$$

$$B(t) = \int_0^t \beta(a)\pi(a)\varphi(P(t-a))B(t-a)da + \int_t^\infty \beta(a)p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da. \tag{20}$$

Now, we define the following functions:

$$G(t) = \varphi(P(t))B(t), \tag{21}$$

$$h_1(t) = \int_t^{\infty} p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da, \tag{22}$$

$$h_2(t) = \int_t^{\infty} \beta(a)p_0(a-t)\frac{\pi(a)}{\pi(a-t)}da. \tag{23}$$

From equations (19)-(23), we obtain that $G(t)$ satisfies the following:

$$G(t) = \varphi\Big(\int_0^t G(t-a)\pi(a)da + h_1(t)\Big)\Big[\int_0^t G(t-a)\beta(a)\pi(a)da + h_2(t)\Big]. \tag{24}$$

We note that $\lim_{t\to+\infty} h_1(t) = \lim_{t\to+\infty} h_2(t) = 0$, because by (2) $\mu(a)$ satisfies $\int_0^{\infty}\mu(a)da = +\infty$.

In the next theorem, we show that either $G(t) \equiv 0$ or $G(t)$ is eventually positive.

Theorem 4.1: Suppose that $\beta(a) > 0$, for $a \in [b,c], b < c$, then:

(1) If $h_2(t) = 0, \forall t \in [0,\infty)$, then $G(t) \equiv 0$,
(2) If $h_2(t) > 0$, for some $t \in [0,\infty)$, then $G(t)$ is eventually positive.

Proof: To prove (1), we suppose that $h_2(t) \equiv 0$, then using equation (24), we obtain

$$\begin{aligned} G(t) &= \varphi\Big(\int_0^t G(t-a)\pi(a)da + h_1(t)\Big)\int_0^t G(t-a)\beta(a)\pi(a)da \\ &\leq \varphi(0)\int_0^t G(t-a)\beta(a)\pi(a)da \\ &\leq \varphi(0)||\beta||_{\infty}\int_0^t G(a)da. \end{aligned}$$

Therefore, $G(t) = 0$, by Gronwall's inequality. This completes the proof of (1).

To prove (2), we note that if $h_2(t) > 0$, for some $t \in [0,\infty)$, then $G(t) > 0$, because φ is positive. Now, we assume that $G(t) > 0$, for $t \in [\alpha,\gamma]$, where $0 \leq \alpha < \gamma$. Then for $t \in (\alpha + b, \gamma + c)$, we obtain

$$\int_0^t G(t-a)\beta(a)\pi(a)da \geq \min_{t\in[\alpha,\gamma]} G(t)\int_{(t-\gamma)_+}^{t-\alpha}\beta(a)\pi(a)da > 0,$$

where $(t-\gamma)_+$ is defined as

$$(t-\gamma)_+ = \begin{cases} t-\gamma, & t > \gamma, \\ 0, & t \leq \gamma, \end{cases}$$

also note that the last inequality is because $[b,c] \cap \Big((t-\gamma)_+, t-\alpha\Big) \neq \phi$.

Accordingly, we assume that $G(t) > 0$, for $t \in [\alpha + nb, \gamma + nc]$, and consider $t \in (\alpha + (n+1)b, \gamma + (n+1)c)$ to obtain

$$\int_0^t G(t-a)\beta(a)\pi(a)da \geq \min_{t\in[\alpha+nb,\gamma+nc]} G(t) \int_{(t-(\gamma+nc))_+}^{t-(\alpha+nb)} \beta(a)\pi(a)da > 0.$$

Therefore, $G(t) > 0$, for $t \in [\alpha + (n+1)b, \gamma + (n+1)c]$. Finally, we note that $\exists\ n_0$ such that $\alpha + (n+1)b < \gamma + nc$, $\forall n \geq n_0$ and hence $\exists\ t_0 \geq 0$ such that $\forall\ t \geq t_0$, we have $[t, \infty) \subset \cup_{n=1}^{\infty}[\alpha + nb, \gamma + nc]$, and therefore, $G(t)$ is eventually positive. This completes the proof of (2), and therefore, the proof of the theorem is completed.

We note that in Iannelli (1995), a similar proof to Theorem 4.1 (2) is given.

In the next theorem, we prove the global stability of the trivial steady state. We note that in Theorem 3.1, we have shown that the trivial steady state is locally asymptotically stable if $\varphi(0) < R_0$.

Theorem 4.2: The trivial steady state is globally stable if $\varphi(0) < R_0$.

Proof: Suppose that $\varphi(0) < R_0$, and let $G^{\infty} = \limsup_{t\to+\infty} G(t)$. Then from equation (24) and Fatou's Lemma, we obtain $G^{\infty} \leq G^{\infty}\varphi(0)\int_0^{\infty} \beta(a)\pi(a)da = G^{\infty}\frac{\varphi(0)}{R_0} < G^{\infty}$, which is impossible unless $G^{\infty} = 0$. Therefore, the trivial steady state is globally stable if $\varphi(0) < R_0$. This completes the proof of the theorem.

To facilitate our further study of global stability, we define two functions f_1 and f_2 and note that f_1 is similar to that defined in Iannelli, et al. (2002).

$$f_1(x) = \begin{cases} P_{\infty}\varphi(P_{\infty})\dfrac{[1 + \frac{\varphi(P_{\infty})}{\varphi(0)}]}{[x + P_{\infty}\frac{\varphi(P_{\infty})}{\varphi(0)}]}, & x \in [0, P_{\infty}], \\ \dfrac{(x - P_{\infty})}{P_{\infty}}[\varphi(2P_{\infty}) - \varphi(P_{\infty})] + \varphi(P_{\infty}), & x \in [P_{\infty}, \tau], \\ \dfrac{f_1(i\tau)}{[1 + \alpha_i(x - i\tau)]}, \quad x \in (i\tau, (i+1)\tau], & i = 1, 2, 3, ..., \end{cases} \tag{25}$$

where

$$\begin{aligned} \tau &= \sup_{x\in(P_{\infty},2P_{\infty}]} \{x : (xf_1(x))' \geq 0\}, \\ \alpha_i &= \frac{1}{\tau}\min\Big\{\frac{1}{i}, \frac{f_1(i\tau)}{\varphi(i\tau)} - 1\Big\}, \quad i \geq 1, \end{aligned}$$

$$f_2(x) = \begin{cases} \epsilon\varphi'(P_{\infty})(x - P_{\infty}) + \varphi(P_{\infty}), & x \in [0, P_{\infty}], \\ \varphi(x), & x \in [P_{\infty}, \infty), \end{cases} \tag{26}$$

where $0 < \epsilon < \min\Big\{-\dfrac{\varphi(P_{\infty})}{P_{\infty}\varphi'(P_{\infty})}, 1\Big\}$.

We note that $\tau > P_\infty$, since $(xf_1(x))'|_{x=P_\infty} = \varphi(2P_\infty) > 0$.

We also, note that for $i \geq 2$, we obtain $f_1(i\tau) = \dfrac{f_1((i-1)\tau)}{[1+\alpha_{i-1}\tau]} \geq \varphi((i-1)\tau) > \varphi(i\tau)$, and accordingly $\alpha_i \geq 0$, for $i = 1, 2, ...$, and hence f_1 is non-increasing, and satisfies $(xf_1(x))' \geq 0,\ \forall x \in [0,\infty)$, $f_1(i\tau) \geq \varphi(i\tau)$, for $i = 1, 2, ...$, $f_1'(x) < 0$, for $x \in [0, \tau]$, and $f_1(P_\infty) = \varphi(P_\infty)$.

For f_2, we note that ϵ satisfies $0 < \epsilon < 1$, and therefore, $(xf_2(x))' \geq 0,\ \forall x \in [0, P_\infty]$, $f_2(P_\infty) = \varphi(P_\infty)$, $f_2'(x) < 0$, for $x \in [0, P_\infty]$, and $f_2(x) \leq \varphi(x),\ \forall x \in [0,\infty)$.

In order to obtain the global stability result for the nontrivial steady state, we confine our study of problem (1) to the case when individuals in the population can only attain a maximum finite age $A > 0$ and $\beta(a)$ takes the following form:

$$\beta(a) = \begin{cases} \beta = constant, & a \in [0, A], \\ 0, & a \notin [0, A]. \end{cases} \tag{27}$$

We note that in this case $G(t)$ satisfies the following for $t > A$:

$$G(t) = \beta\varphi\Big(\int_0^A G(t-a)\pi(a)da\Big)\int_0^A G(t-a)\pi(a)da. \tag{28}$$

In the next result, we use Theorem 4.1 to prove that $\liminf_{t\to+\infty} G(t) > 0$. The proof of this result is similar to that given in Iannelli (1995), though we have constructed a function f_2 with the desired monotonicity properties instead of assuming these properties.

To facilitate our writing, we define the following:

$$\begin{aligned} I_n &= [nA, (n+1)A],\ n \geq 0, \\ m_n &= \min_{t\in I_n} G(t), \quad \tilde{m}_n = \min\{m_n, m_{n-1}\}, \\ M_n &= \max_{t\in I_n} G(t), \quad \tilde{M}_n = \max\{M_n, M_{n-1}\}. \end{aligned}$$

Lemma 4.3: Suppose that $\beta(a)$ satisfies (27) and $h_2(t)$ is not identically zero for $t \in [0,\infty)$, then $\liminf_{t\to\infty} G(t) > 0$.

Proof: Let n_0 be large enough so that by Theorem 4.1, $G(t)$ is positive in $[nA, (n+1)A]$, $\forall n \geq n_0$. Now, let $K = \min\{m_{n_0}, G_\infty\} > 0$ by Theorem 4.1, and suppose by way of contradiction that $m_{n_0+1} < K$. By using equations (28) and (26), we obtain

$$\begin{aligned} m_{n_0+1} &\geq \beta f_2\Big(m_{n_0+1}\int_0^A \pi(a)da\Big)m_{n_0+1}\int_0^A \pi(a)da \\ &> \beta f_2\Big(G_\infty\int_0^A \pi(a)da\Big)m_{n_0+1}\int_0^A \pi(a)da \\ &= m_{n_0+1}, \end{aligned}$$

which is impossible, and accordingly, $m_{n_0+1} \geq K$. Now, an obvious induction argument will complete the proof. This completes the proof of the lemma.

In the next theorem, we prove the global stability of the nontrivial steady state for problem (1) under assumption (27).

Theorem 4.4: Suppose that $\varphi(0) > R_0$ and $\beta(a)$ satisfies (27), then the nontrivial steady state is globally stable.

Proof: From equation (28), we obtain

$$\begin{aligned} G(t) &= \beta\varphi\Big(\int_0^A G(t-a)\pi(a)da\Big)\int_0^A G(t-a)\pi(a)da \\ &\leq \beta f_1\Big(\tilde{M}_n\int_0^A \pi(a)da\Big)\tilde{M}_n\int_0^A \pi(a)da. \end{aligned} \tag{29}$$

where f_1 is given by (25). Also, note that for $a \in [0, A]$, we have $t - a \in I_n \cup I_{n-1}$, and accordingly,

$$M_{n+1} \leq \beta f_1\Big(\tilde{M}_n\int_0^A \pi(a)da\Big)\tilde{M}_n\int_0^A \pi(a)da. \tag{30}$$

In the sequel we prove the following:

(1) If $M_n \leq G_\infty$, then $M_{n+1} \leq G_\infty$,
(2) If $M_n > G_\infty$, then $M_{n+1} < M_n$, and $\lim\limits_{n\to+\infty} M_n = G_\infty$,
(3) If $m_n \geq G_\infty$, then $m_{n+1} \geq G_\infty$,
(4) If $m_n < G_\infty$, then $m_{n+1} > m_n$, and $\lim\limits_{n\to+\infty} m_n = G_\infty$.

To prove (1), suppose that $M_n \leq G_\infty$, and $M_{n+1} > G_\infty$, and hence $M_{n+1} > M_n$, and therefore, by (30), we obtain

$$\begin{aligned} M_{n+1} &\leq \beta f_1\Big(M_{n+1}\int_0^A \pi(a)da\Big)M_{n+1}\int_0^A \pi(a)da \\ &< \beta f_1\Big(G_\infty\int_0^A \pi(a)da\Big)M_{n+1}\int_0^A \pi(a)da \\ &= M_{n+1}, \end{aligned}$$

which is impossible, and so, $M_{n+1} \leq G_\infty$. This completes the proof of (1).

To prove (2), suppose that $M_n > G_\infty$, and $M_{n+1} \geq M_n$, and therefore, by (30),

$$\begin{aligned} M_{n+1} &\leq \beta f_1\Big(M_{n+1}\int_0^A \pi(a)da\Big)M_{n+1}\int_0^A \pi(a)da \\ &< \beta f_1\Big(G_\infty\int_0^A \pi(a)da\Big)M_{n+1}\int_0^A \pi(a)da \\ &= M_{n+1}, \end{aligned}$$

which is impossible, and so, $M_{n+1} < M_n$.

Now, letting $n \to +\infty$ and $M_\infty = \lim_{n\to+\infty} M_n \geq G_\infty$, we obtain

$$\begin{aligned} M_\infty &\leq \beta f_1\Big(M_\infty \int_0^A \pi(a)da\Big) M_\infty \int_0^A \pi(a)da, \\ 1 &\leq \beta f_1\Big(M_\infty \int_0^A \pi(a)da\Big) \int_0^A \pi(a)da. \end{aligned}$$

So, if $M_\infty > G_\infty$, we obtain

$$1 < \beta f_1\Big(G_\infty \int_0^A \pi(a)da\Big) \int_0^A \pi(a)da = 1,$$

which is impossible, and therefore, $M_\infty = G_\infty$. This completes the proof of the (2).

To prove (3), suppose that $m_n \geq G_\infty$, and $m_{n+1} < G_\infty$, and therefore, we obtain

$$\begin{aligned} m_{n+1} &\geq \beta f_2\Big(\tilde{m}_{n+1} \int_0^A \pi(a)da\Big)\tilde{m}_{n+1} \int_0^A \pi(a)da \\ &= \beta f_2\Big(m_{n+1} \int_0^A \pi(a)da\Big)m_{n+1} \int_0^A \pi(a)da \\ &> \beta f_2\Big(G_\infty \int_0^A \pi(a)da\Big)m_{n+1} \int_0^A \pi(a)da \\ &= m_{n+1}, \end{aligned}$$

which is impossible, and accordingly, $m_{n+1} \geq G_\infty$. This completes the proof of (3).

To prove (4), suppose that $m_n < G_\infty$, and $m_{n+1} \leq m_n$, and therefore, we obtain

$$\begin{aligned} m_{n+1} &\geq \beta f_2\Big(\tilde{m}_{n+1} \int_0^A \pi(a)da\Big)\tilde{m}_{n+1} \int_0^A \pi(a)da \\ &> \beta f_2\Big(G_\infty \int_0^A \pi(a)da\Big)m_{n+1} \int_0^A \pi(a)da \\ &= m_{n+1}, \end{aligned}$$

which is impossible, and accordingly, $m_{n+1} > m_n$.

Now, let $n \to +\infty$ and $m_\infty = \lim_{n\to+\infty} m_n \leq G_\infty$, we obtain

$$\begin{aligned} m_\infty &\geq \beta f_2\Big(m_\infty \int_0^A \pi(a)da\Big)m_\infty \int_0^A \pi(a)da, \\ 1 &\geq \beta f_2\Big(m_\infty \int_0^A \pi(a)da\Big) \int_0^A \pi(a)da. \end{aligned}$$

So, if we suppose that $m_\infty < G_\infty$, we obtain

$$1 > \beta f_2\Big(G_\infty \int_0^A \pi(a)da\Big) \int_0^A \pi(a)da = 1,$$

which is impossible, and therefore, $m_\infty = G_\infty$. This completes the proof of the (4).

Now, the proof of the theorem can be completed by using (1)-(4) and the Sandwich Theorem. This completes the proof of the theorem.

5. Generalization of the Model

In this section, we consider the following related model which is considered in Iannelli (1995).

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial a} + \dfrac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0,\ a > 0,\ t > 0, \\ p(0,t) = \tilde{R}_0 \varphi(S(t))B(t),\ t \geq 0, \\ p(a,0) = p_0(a),\ a \geq 0, \\ S(t) = \displaystyle\int_0^A \gamma(a)p(a,t)da,\ t \geq 0, \\ B(t) = \displaystyle\int_0^A \beta(a)p(a,t)da,\ t \geq 0. \end{cases} \tag{31}$$

We assume that $p(a,t)$ is as before, $\mu(a)$ and $\beta(a)$ are as before and satisfy (2) and that $\int_0^A \beta(a)\pi(a)da = 1$ where A is the maximum age that an individual in the population can attain, $0 \leq \gamma(a) \in L_\infty[0,A]$, $0 \leq \tilde{R}_0$ is a constant. Also, we assume that $\varphi(t)$ satisfies (3) as well as $\varphi(0) = 1$.

We note that similar calculations to that given in section 2 yield the following result.

Theorem 5.1:

(1) Suppose that $\tilde{R}_0 \leq 1$, then the trivial steady state $p_\infty(a) \equiv 0$ is the only steady state.
(2) Suppose that $\tilde{R}_0 > 1$, then the trivial steady state as well as the nontrivial steady state are possible steady states. The nontrivial steady state is unique when it exists.

For the local stability results, we note that similar calculations to that given in section 3 yield the following characteristic equation:

$$1 = \int_0^A e^{-\lambda a}\pi(a)[\beta(a) + \tilde{R}_0 B_\infty \varphi'(P_\infty)\gamma(a)]da. \tag{32}$$

For the stability of the trivial steady state, we obtain the following result.

Theorem 5.2: The trivial steady state is locally stable whenever it exists.

We note that, for the proof of Theorem 5.2, the characteristic equation in this case becomes

$$1 = \int_0^A e^{-\lambda a}\pi(a)\beta(a)da, \tag{33}$$

and therefore, if we suppose that $Re\lambda \geq 0$, then $\lambda = 0$ is the only possible solution of (33). And therefore, the trivial steady state is locally stable.

Concerning the local stability of the nontrivial steady state, we note that the following result gives four sufficient conditions for the local asymptotic stability, the proofs of these results are similar to those of Theorems 3.2-3.5, and therefore are omitted.

Theorem 5.3: Suppose that $\tilde{R}_0 > 1$, then the nontrivial steady state is locally asymptotically stable in each of the following cases:

1) $\mu^2(a) > \mu'(a),\ \forall a \in [0, A]$,
2) $\beta(a) + \tilde{R}_0\varphi'(S_\infty)B_\infty\gamma(a) \geq 0,\ \forall a \in [0, A]$,
3) $\displaystyle\int_0^A \pi(a)\Big|\beta(a) + \tilde{R}_0\varphi'(S_\infty)B_\infty\gamma(a)\Big|da < 1$,
4) $\beta(a)$ and $\gamma(a)$ are constants independent of age a.

For the global stability of the trivial and the nontrivial steady states, we note that in this case $G(t) = \varphi(S(t))B(t)$ satisfies the following for $t > A$:

$$G(t) = \tilde{R}_0\varphi\Big(\int_0^A G(t-a)\gamma(a)\pi(a)da\Big)\int_0^A G(t-a)\beta(a)\pi(a)da. \tag{34}$$

The proofs of the following two theorems are similar to the proofs of Theorems 4.2, 4.4, respectively, and therefore are omitted.

Theorem 5.4: The trivial steady state is globally stable if $\tilde{R}_0 < 1$.

Theorem 5.5: Suppose that $\tilde{R}_0 > 1$ and $\beta(a) = constant.\gamma(a)$, then the nontrivial steady state is globally stable.

We note that Theorem 5.5 is proved in Iannelli (1995), under the additional assumption that $t \rightarrow t\varphi(t)$ is nondecreasing.

6. Conclusion

In this paper, we studied an age-structured population model with cannibalism that has been studied in Bekkal-Brikci, et al. (2007), and in a more general form in Rorres (1976), and Iannelli (1995). We refer the reader to the above mentioned references and the references therein for background, modeling, and the motivation of the problem.

We determined the steady states of the model and examined their local and global stability.

We proved that if $\varphi(0) \leq R_0$, then the only steady state is the trivial state, and if $\varphi(0) > R_0$, then a trivial steady state as well as a nontrivial steady state, which is unique when it exists, are possible steady states.

We showed that the trivial steady state is globally stable when $\varphi(0) < R_0$, and is unstable when $\varphi(0) > R_0$. We also gave several sufficient conditions for the local asymptotic stability of the nontrivial steady state.

Moreover, we proved the global stability of the nontrivial steady state when the individuals in the population are assumed to have a maximum finite age and the birth rate is a constant independent of age.

The above results generalize those given in Bekkal-Brikci, et al. (2007) in that they give results for the global stability of the age-dependent model, also we give other three sufficient conditions for the local asymptotic stability of the nontrivial steady state, and our proof for the sufficient condition for the local asymptotic stability of the nontrivial steady state which is given in Bekkal-Brikci, et al. (2007), and also in Iannelli (1995), is shorter and straightforward.

We also studied the general model given in Rorres (1976), and Iannelli (1995), and showed that our results in this paper generalize those given therein.

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