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Remarks on the Stability of Some Size-Structured Population Models V: The Case When the Death Rate Depends on Adults Only and the Growth Rate Depends on Size Only

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Abstract

We continue our study of size-structured population dynamics models when the population is divided into adults and juveniles, started in El-Doma (To appear). We concentrate our efforts in the special case when the death rate depends on adults only, the growth rate depends on size only and the maximum size for an individual in the population is infinite. Three demographic parameters are identified and are shown to determine conditions for the (in)stability of a nontrivial steady state. We also give examples that illustrate the stability results. The results in this paper generalize previous results, for example, see Calsina, et al. (2003), El-Doma (2006), and El-Doma (2008).

Keywords: Adults; Juveniles; Population; Size-structure; Stability; Steady State

MSC (2000) #: 45M10; 35B35; 35L60; 92D25

1. Introduction

IN this paper, we continue our study of a size-structured population dynamics model that divides the population at any time t into adults with size larger than the maturation size $T \geq 0$, we denote by $A(t)$, and juveniles with size smaller than the maturation size, we denote by $J(t)$, started in El-Doma (To appear). The vital rates i.e., the birth rate, the death rate, and the growth rate, depend on size, adults, and juveniles, accordingly, the model takes into account the limited resources as well as the intra-specific competition between adults and juveniles.

In this paper, we concentrate our efforts in the study of the special case when the death rate depends on adults only, the growth rate depends on size only, and the maximum size for an individual in the population is infinite. The motivation for assuming that the death rate depends on adults only is that almost all species protect their young (juveniles) by sheltering and caring, though this is species specific. Also when disturbed or attacked by predators, for example, some females even take their young into their mouth, for example, see Taborsky (2006). This assumption will also allow us to generalize stability results given, for example, in Gurney, et al. (1980) and Weinstock, et al. (1987) for the classical age-structured population dynamics model of Gurtin, et al. (1974), which corresponds to problem (1.1) in El-Doma (To appear) when $V \equiv 1$, and $T = 0$.

We study the stability of the nontrivial steady states given by Theorem 2.1 (2) in El-Doma (To appear). We identify three demographic parameters that determine the (in)stability of a nontrivial steady state. We also obtain several conditions for the (in)stability of a nontrivial steady state via these demographic parameters. We also give examples that illustrate our stability results.

In the last paper of our series, further stability results will be given for the case when, $V(a, J, A) = V(a), \mu(a, J, A) = \mu(J)$, and the case when, $V(a, J, A) = V(a), \mu(a, J, A) = \mu(a)$.

The organization of this paper as follows: in section 2 we obtain stability results, and give examples that illustrate some of our theorems; in section 3 we conclude our results.

2. Stability of the Nontrivial Steady States

In the following, we obtain stability results for the special case when, $l = +\infty, V(a, J, A) = V(a)$, $\mu(a, J, A) = \mu(A)$, and $\displaystyle\int_0^\infty \frac{d\tau}{V(\tau)} = +\infty$.

We note that if $\mu(A_\infty) = 0$, then from equation (4) in El-Doma (to appear), we obtain that $P_\infty = +\infty$. Therefore, we assume that, $\mu(A_\infty) > 0$, throughout the paper.

We also note that, in this case, Corollary 3.7 in El-Doma (to appear), gives the following condition for a nontrivial steady state to be locally asymptotically stable:

$$\int_T^l \frac{e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}\Big|\Big[\beta(a, J_\infty, A_\infty) + \delta\Big]\Big|da + \tag{1}$$
$$|\gamma|\int_0^T \frac{e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}da + \int_T^l\int_0^a F(a,\sigma)\Big|g_A(\sigma, J_\infty, A_\infty)\Big|d\sigma da$$

$$+\int_T^l \int_T^l \int_0^a \frac{e^{-\mu(A_\infty)\int_0^b \frac{d\tau}{V(\tau)}}}{V(b)} F(a,\sigma) \Big| g_A(\sigma, J_\infty, A_\infty)\Big[\beta(b, J_\infty, A_\infty) - \beta(a, J_\infty, A_\infty)\Big]\Big| d\sigma da db$$
$$+|\gamma| \int_0^T \int_T^l \int_0^a \frac{e^{-\mu(A_\infty)\int_0^b \frac{d\tau}{V(\tau)}}}{V(b)} F(a,\sigma)\Big| g_A(\sigma, J_\infty, A_\infty)\Big| d\sigma da db +$$
$$|\gamma| \int_T^l \int_0^T \int_0^a \frac{e^{-\mu(A_\infty)\int_0^b \frac{d\tau}{V(\tau)}}}{V(b)} F(a,\sigma)\Big| g_A(\sigma, J_\infty, A_\infty)\Big| d\sigma da db < 1,$$

where δ, and γ are given, respectively, by equations (22), (23) in El-Doma (to appear).

Also from Theorem 3.2 in El-Doma (to appear), we obtain the following condition for the instability of a nontrivial steady state:

$$\delta A_\infty + \gamma J_\infty + \mu'(A_\infty)\Big\{\gamma\Big[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da\Big]$$
$$-A_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty)\frac{p_\infty(a)}{V(\sigma)} d\sigma da\Big\} > 0. \tag{2}$$

Also, in this case, by straightforward integrations in the characteristic equation (11) in El-Doma (to appear), we obtain

$$1 = \frac{1}{V(0)}\int_T^l \beta(a, J_\infty, A_\infty) e^{-\int_0^a E(\tau)d\tau} da + \frac{\xi e^{-[\xi+\mu(A_\infty)]m}}{[\xi+\mu(A_\infty)][\xi+\chi]}\delta$$
$$+\frac{1}{[\xi+\mu(A_\infty)]}\Big\{1 - \frac{e^{-[\xi+\mu(A_\infty)]m}\Big[\xi + e^{\mu(A_\infty)m}\chi\Big]}{[\xi+\chi]}\Big\}\gamma,\ \xi \neq 0, \tag{3}$$

where χ, and m are given by

$$\chi = \frac{p_\infty(0)V(0)\mu'(A_\infty)e^{-\mu(A_\infty)m}}{\mu(A_\infty)}, \tag{4}$$
$$m = \int_0^T \frac{d\tau}{V(\tau)}. \tag{5}$$

The stability results that we are going to obtain are in terms of the following three demographic parameters:

$$\delta = \int_T^\infty \beta_A(a, J, A)p_\infty(a)da,$$
$$\gamma = \int_T^\infty \beta_J(a, J, A)p_\infty(a)da,$$
$$\chi = \frac{p_\infty(0)V(0)\mu'(A_\infty)e^{-\mu(A_\infty)m}}{\mu(A_\infty)},$$
$$= P_\infty \mu'(A_\infty)e^{-\mu(A_\infty)m}.$$

We note that δ can be interpreted as the total change in the birth rate, at the steady state, due to a change in adults only. Also, note that γ can be interpreted similarly.

If $T = 0$, then $\chi = P_\infty \mu'(P_\infty)$ and therefore, it can be interpreted as the total change in the death rate, at the steady state, due to a change in the population, for example, see Weinstock, et al. (1987). If $T \neq 0$, then χ can be interpreted as the total change in the death rate, at the steady state, due to a change in adults only. Note that the factor $e^{-\mu(A_\infty)m}$ in the formula defining χ when $T \neq 0$, is the probability of survival up size T.

We expect that $\delta < 0, \gamma < 0$, and $\chi \geq 0$ are conditions that imply the local asymptotic stability of a nontrivial steady state, for example, see El-Doma (2008), for the special case when, $T = 0$. On the other hand, from (2), it is easy to see that if $\delta > 0, \gamma > 0$, and $\chi = 0$, then a nontrivial steady state is unstable.

In the next result, we describe the stability of a nontrivial steady state when, $T = 0$. This special case is proved in El-Doma (2008), and therefore, the proof is omitted.

Theorem 2.1 Suppose that, $T = 0, \chi \geq 0$, and, $\delta \leq 0$, with both not equal to zero. Then a nontrivial steady state is locally asymptotically stable.

We note that Theorem 2.1 is important for our further stability results since it establishes the local asymptotic stability of a nontrivial steady state when, $T = 0$.

In the following result, we describe the stability of a nontrivial steady state in the special case when, $\delta = \gamma = 0$.

Theorem 2.2 Suppose that, $\delta = \gamma = 0$. Then a nontrivial steady state is locally asymptotically stable if $\chi > 0$, and, unstable if $\chi < 0$.

Proof. We note that in this case, the characteristic equation (3) can be rewritten in the following form:

$$\left[1 + \frac{\chi}{\xi}\right]\left[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a, J\infty, A_\infty) e^{-\xi \int_0^a \frac{d\tau}{V(\tau)}} da\right] = 0, \quad \xi \neq 0. \tag{6}$$

From equation (6), we see that if $\chi < 0$, then $\xi = -\chi > 0$ is a root of equation (6), and therefore, we obtain instability.

On the other hand, if $\chi > 0$, then $\xi = -\chi < 0$, is a root of equation (6), and the only other possible root of equation (6) is when

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} \pi(a, J\infty, A_\infty) e^{-\xi \int_0^a \frac{d\tau}{V(\tau)}} da = 0, \quad \xi \neq 0. \tag{7}$$

Now, suppose that $\xi = x + iy, x \geq 0$, then by equation (6) in El-Doma (to appear), it is easy to see that the only possible root of equation (7) is, $\xi = 0$, and Theorem 3.3 in El-Doma (to appear) shows that, $\xi = 0$, is not a root of the characteristic equation (11) in El-Doma (to appear) since, $\chi > 0$. According, a nontrivial steady state is locally asymptotically stable if $\chi > 0$. This completes the proof of the theorem.

In the next result, we show that a nontrivial steady state is unstable if $\chi < 0$, and, $\delta = (1 - e^{\mu(A_\infty)m})\gamma$.

Theorem 2.3 Suppose that, $\chi < 0$, and, $\delta = (1 - e^{\mu(A_\infty)m})\gamma$. Then a nontrivial steady state is unstable.

Proof. We note that, in this case, it is easy to see that $\xi = -\chi > 0$ is a root of the characteristic equation (3). This completes the proof of the theorem.

We note that in Theorem 2.3, if we set $\gamma = 0$, then $\delta = 0$, and hence we retain the result of Theorem 2.2.

We note that according to Theorem 3.2 in El-Doma (to appear), a nontrivial steady state is unstable i.e., $\xi > 0$, is a root of the characteristic equation (3) if $\Xi > 0$. Also by Theorem 3.3 in El-Doma (to appear) if $\Xi = 0$, then, $\xi = 0$, is a root of the characteristic equation (3). Therefore, a necessary condition for a nontrivial steady state to be hyperbolic and locally asymptotically stable is, $\Xi < 0$.

In the following lemma, we give sufficient conditions for, Ξ, to be negative.

Lemma 2.4 Suppose that, $l = +\infty, V(a, J, A) = V(a), \mu(a, J, A) = \mu(A), \chi \geq 0$, $R_J(J_\infty, A_\infty) < 0$, and, $R_A(J_\infty, A_\infty) < 0$. Then

$$\Xi = \Big[(1 + G_A(T, l, 0))J_\infty - G_A(0, T, 0)A_\infty\Big]R_J(J_\infty, A_\infty) + \Big[(1 + G_J(0, T, 0))A_\infty - G_J(T, l, 0)J_\infty\Big]R_A(J_\infty, A_\infty) < 0. \quad (8)$$

Proof. We note that, in this case, $G_J(T, l, 0) = G_J(0, T, 0) = 0$, and therefore, we only need to show that

$$\Big[(1 + G_A(T, l, 0))J_\infty - G_A(0, T, 0)A_\infty\Big]R_J(J_\infty, A_\infty) + A_\infty R_A(J_\infty, A_\infty) < 0. \quad (9)$$

Now, we notice that by using the Mean Value Theorem, we obtain $G_A(T, l, 0)J_\infty - G_A(0, T, 0)A_\infty = \mu'(A_\infty)J_\infty A_\infty\Big[\int_0^{y_1}\frac{d\tau}{V(\tau)} - \int_0^{y_2}\frac{d\tau}{V(\tau)}\Big] \geq 0$, for $y_1 \in [T, l]$ and $y_2 \in [0, T]$ since $\chi \geq 0$ implies that $\mu'(A_\infty) \geq 0$. Hence (9) follows immediately. This completes the proof of the lemma.

We note that from the characteristic equation (3), if we let $\xi = x + iy$, then we obtain the following pair of equations:

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[x+\mu(A_\infty)]\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{e^{-[x+\mu(A_\infty)]m}}{\Delta}[\delta - \gamma] \times \Big\{[y^2(x + \mu(A_\infty) + \chi) + x(x + \mu(A_\infty))(\chi + x)] \cos ym + y[\mu(A_\infty)\chi - (x^2 + y^2)] \sin ym\Big\} + \frac{\gamma}{\Delta}\Big\{(x + \mu(A_\infty))(x + \chi)^2 + y^2(x + \mu(A_\infty)) + [\chi y^2 - \chi(x + \mu(A_\infty))(x + \chi)]e^{-xm} \cos ym + [\chi y(x + \mu(A_\infty) + x + \chi)e^{-xm}] \sin ym\Big\}, \quad (10)$$

$$\int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-[x+\mu(A_\infty)]\int_0^a \frac{d\tau}{V(\tau)}} \sin y \int_0^a \frac{d\tau}{V(\tau)} da = \delta\frac{e^{-[x+\mu(A_\infty)]m}}{\Delta} \times \Big\{[\mu(A_\infty)y\chi - y(x^2 + y^2)] \cos ym - [(x(x + \mu(A_\infty)) + y^2)(x + \chi) + \mu(A_\infty)y^2] \sin ym\Big\}$$

$$+\frac{\gamma}{\Delta}\Big\{-y^3 - y[x+\chi]^2 + [x^2 y e^{-[x+\mu(A_\infty)]m} - \mu(A_\infty) y\chi e^{-[x+\mu(A_\infty)]m} + y^3 e^{-[x+\mu(A_\infty)]m}$$
$$+y\chi(x+\mu(A_\infty)+x+\chi)e^{-xm}]\cos ym + [\chi e^{-xm}(x+\mu(A_\infty)(x+\chi) - \chi y^2 e^{-xm} +$$
$$x(x+\mu(A_\infty))(x+\chi)e^{-[x+\mu(A_\infty)]m} + y^2 e^{-[x+\mu(A_\infty)]m}(\mu(A_\infty)+x+\chi)]\sin ym\Big\}, \tag{11}$$

where Δ is given by

$$\Delta = [(x+\mu(A_\infty))^2 + y^2][(x+\chi)^2 + y^2]. \tag{12}$$

We note that the following conditions are for crossing the imaginary axis, for example, see Thieme, et al. (1993), and Iannelli (1995), stem from the fact that by Theorem 2.1 if $T = 0, \chi \geq 0$, and, $\delta \leq 0$, with both not equal to zero, then all the roots of the characteristic equation lie to the left of the imaginary axis, and by further conditions, for example, see Lemma 2.4, they can only cross the imaginary axis to the right-half plane as T increases by crossing the imaginary axis when, $y \neq 0$:

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{e^{-\mu(A_\infty)m}}{\Delta_0}[\delta - \gamma] \times$$
$$\Big\{y^2[\mu(A_\infty)+\chi]\cos ym + y[\mu(A_\infty)\chi - y^2]\sin ym\Big\} + \frac{\gamma}{\Delta_0}\Big\{\mu(A_\infty)\chi^2$$
$$+\mu(A_\infty)y^2 + [\chi y^2 - \mu(A_\infty)\chi^2]\cos ym + \chi y[\mu(A_\infty)+\chi]\sin ym\Big\}, \tag{13}$$

$$\int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \sin y \int_0^a \frac{d\tau}{V(\tau)} da = [\delta - \gamma]\frac{e^{-\mu(A_\infty)m}}{\Delta_0} \times$$
$$\Big\{[\mu(A_\infty)y\chi - y^3)]\cos ym - y^2[\chi + \mu(A_\infty)]\sin ym\Big\} +$$
$$\frac{\gamma}{\Delta_0}\Big\{-y^3 - y\chi^2 + y\chi[\mu(A_\infty)+\chi]\cos ym + \chi[\chi\mu(A_\infty) - y^2]\sin ym\Big\}, \tag{14}$$

where Δ_0 is define by

$$\Delta_0 = [\mu(A_\infty)^2 + y^2][\chi^2 + y^2]. \tag{15}$$

In the next result, we describe the stability of a nontrivial steady state when, $\delta = \gamma < 0 = \chi$.

Theorem 2.5 Suppose that, $\delta = \gamma < 0 = \chi$. Then a nontrivial steady state is locally asymptotically stable.

Proof. In this case, from equations (13) and (15), we obtain

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{\gamma\mu(A_\infty)}{[\mu(A_\infty)^2 + y^2]}. \tag{16}$$

We note that the left-hand side of equation (16) is positive by equation (6) in El-Doma (to appear) since $y \neq 0$ by using Lemma 2.4, whereas the right-hand side is negative because $\gamma < 0$. Accordingly, crossing is impossible, and therefore, by Theorem 2.1, a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $\delta = \gamma < 0$, and $\mu(A_\infty) \geq \chi \geq 0$.

Theorem 2.6 Suppose that, $\delta = \gamma < 0$, and, $\mu(A_\infty) \geq \chi \geq 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that, in this case, from equation (13), we obtain

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da = \frac{\gamma}{\Delta_0} \Big\{ \mu(A_\infty)\chi^2 + \mu(A_\infty)y^2 + [\chi y^2 - \mu(A_\infty)\chi^2] \cos ym + \chi y [\mu(A_\infty) + \chi] \sin ym \Big\}. \quad (17)$$

Since by Lemma 2.4, the left-hand side of equation (17) is positive, we obtain

$$\mu(A_\infty)\chi^2 + \mu(A_\infty)y^2 + [\chi y^2 - \mu(A_\infty)\chi^2] \cos ym + \chi y [\mu(A_\infty) + \chi] \sin ym < 0. \quad (18)$$

Suppose that $\chi = \mu(A_\infty)$, and $\cos ym = -1$, then from inequality (18), we obtain $2\mu(A_\infty)^3 < 0$, which is a contradiction, and hence the result follows in this case.

Now, suppose that the above special case does not occur, then from inequality (18), we obtain

$$\begin{aligned} & [\mu(A_\infty) + \chi \cos ym] \Big[y + \frac{\chi(\mu(A_\infty) + \chi) \sin ym}{2(\mu(A_\infty) + \chi \cos ym} \Big]^2 \\ < \quad & \frac{\chi^2(\mu(A_\infty) - \chi)(1 - \cos ym)}{4[\mu(A_\infty) + \chi \cos ym]} [(\mu(A_\infty) - \chi)(1 + \cos ym) - 4\mu(A_\infty)] \\ < \quad & \frac{\chi^2(\mu(A_\infty) - \chi)(1 - \cos ym)}{4[\mu(A_\infty) + \chi \cos ym]} [-2\mu(A_\infty)] \\ \leq \quad & 0, \end{aligned}$$

which is a contradiction, and therefore, a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

We note that alternatively, we can give another proof for Theorem 2.6 as follow.

We note that, in this case, if we let $\xi = x + iy$, and assume that $x \geq 0$, then the characteristic equation (3) takes the following form:

$$\begin{aligned} 1 \quad = \quad & \int_T^l \beta(a, J_\infty, A_\infty)\pi(a, J_\infty, A_\infty) e^{-[x+iy]\int_0^a \frac{d\tau}{V(\tau)}} da + \frac{\gamma[x + \mu(A_\infty) - iy]}{\Big[\Big(x + \mu(A_\infty)\Big)^2 + y^2\Big]} \\ & - \frac{\gamma\chi e^{-(x+iy)m}[x + \chi - iy][x + \mu(A_\infty) - iy]}{[(x+\chi)^2 + y^2]\Big[\Big(x + \mu(A_\infty)\Big)^2 + y^2\Big]}, \quad \xi = x + iy \neq 0. \end{aligned}$$

In order to show that the above characteristic equation does not have a root with, $x \geq 0$, we only need to show that $\frac{[x + \mu(A_\infty)]^2}{\Big[\Big(x + \mu(A_\infty)\Big)^2 + y^2\Big]^2} - \frac{\chi^2 e^{-2xm}}{[(x+\chi)^2 + y^2]\Big[\Big(x + \mu(A_\infty)\Big)^2 + y^2\Big]} \geq 0$, and this is easy to show by straightforward calculation provided that $\mu(A_\infty) \geq \chi$. This completes the proof of the theorem.

We note that the result of Theorem 6 generalizes that of Theorem 2.5.

In the next result, we prove a corollary to Theorem 2.6 which deals with the case when, $\mu(A)$, is a constant.

Corollary 2.7 Suppose that, $\delta = \gamma < 0$, and, $0 < \mu(A)$ is a constant. Then a nontrivial steady state is locally asymptotically stable.

Proof. This result follows directly from Theorem 2.6 since, in this case, $\chi = 0$, and accordingly, $\mu(A_\infty) > \chi$. This completes the proof of the corollary.

In order to generalize Theorem 2.6 to the case, $\chi < 0$, we need to assume the following condition:

$$\delta P_\infty + \frac{\chi e^{\mu(A_\infty)m}}{P_\infty}\Big\{\delta\Big[J_\infty \int_T^l \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da - A_\infty \int_0^T \int_0^a \frac{p_\infty(a)}{V(\sigma)} d\sigma da\Big]$$
$$-A_\infty \int_T^l \int_0^a \beta(a, J_\infty, A_\infty)\frac{p_\infty(a)}{V(\sigma)} d\sigma da\Big\} < 0. \tag{19}$$

In the next result, we will assume condition (19) and obtain a result that generalizes Theorem 2.6 to the case, $\chi < 0$. We note that condition (19) is to assure crossing the imaginary axis with $y \neq 0$.

Theorem 2.8 Suppose that, inequality (19) holds, $\delta = \gamma < 0$, and, $\mu(A_\infty) \geq |\chi|$. Then a nontrivial steady state is locally asymptotically stable.

Proof. Suppose that $\chi = \mu(A_\infty)$, and, $\cos ym = -1$, then from inequality (18), we obtain $2\mu^3(A_\infty) < 0$, which is a contradiction, and hence the result follows in this case. Also, if $\chi = -\mu(A_\infty)$, and $\cos ym = 1$, then from inequality (18), we obtain $0 < 0$, which is a contradiction, and hence the result also follows in this case.

We note that the remaining part of the proof follows the same arguments as in Theorem 2.6 to conclude that the roots of the characteristic equation can not cross the imaginary axis. The result is completed by observing that if $\chi = 0$, then by Theorem 2.6 the result holds. This completes the proof of the theorem.

We note that by arguments similar to that used in Lemma 2.4, we can show that if $\chi \geq 0$, and $\delta = \gamma < 0$, then inequality (19) is automatically satisfied, and accordingly, we obtain Theorem 2.6 from Theorem 2.8.

In the next result, we describe the stability of a nontrivial steady state when, $[e^{\mu(A_\infty)m} + 1]\gamma < \delta < 0, \delta \neq \gamma$, and $\chi = 0$.

Theorem 2.9 Suppose that, $[e^{\mu(A_\infty)m} + 1]\gamma < \delta < 0, \delta \neq \gamma$, and, $\chi = 0$. Then a nontrivial steady state is locally asymptotically stable if $|\delta - \gamma|$, is sufficiently large, and, $\delta - \gamma$, have the appropriate sign.

Proof. We note that, $\chi = 0$, implies $\mu'(A_\infty) = 0$, and therefore, by Lemma 2.4 and equation (13), we obtain the following condition for crossing:

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da \tag{20}$$

$$-\frac{\mu(A_\infty)}{[\mu(A_\infty)^2+y^2]}\Big\{\gamma+e^{-\mu(A_\infty)m}[\delta-\gamma]\cos ym\Big\}=-\frac{y\sin ym}{[\mu(A_\infty)^2+y^2]}[\delta-\gamma]e^{-\mu(A_\infty)m}.$$

If $\sin ym=0$, then from equation (20), we obtain

$$1-\int_T^l\frac{\beta(a,J_\infty,A_\infty)}{V(a)}e^{-\mu(A_\infty)\int_0^a\frac{d\tau}{V(\tau)}}\cos y\int_0^a\frac{d\tau}{V(\tau)}da= \frac{\mu(A_\infty)}{[\mu(A_\infty)^2+y^2]}\Big\{\gamma(1-e^{-\mu(A_\infty)m}\cos ym)+\delta e^{-\mu(A_\infty)m}\cos ym\Big\}, \tag{21}$$

and since $\cos ym=\underline{+}1$, it is easy to see that we can obtain a contradiction when $\cos ym=1$ since the left-hand side of equation (21) is positive whereas the right-hand side is negative. If $\cos ym=-1$, then we similarly use the condition $[e^{\mu(A_\infty)m}+1]\gamma<\delta<0$, which also gives a contradiction.

Now, we assume that $\sin ym\neq 0$, and accordingly, we can use equation (20) to solve for y and use it in equation (14) to obtain the following condition for crossing:

$$\frac{\mu(A_\infty)e^{-\mu(A_\infty)m}[\delta-\gamma]}{[\mu(A_\infty)^2+y^2]\sin ym}\Big\{\Big(\sin ym+\frac{[\mu(A_\infty)^2+y^2]}{2\mu(A_\infty)e^{-\mu(A_\infty)m}[\delta-\gamma]}\Big)^2+ \Big[\cos ym+\frac{[\mu(A_\infty)^2+y^2]}{2\mu(A_\infty)e^{-\mu(A_\infty)m}[\delta-\gamma]}\Big(\frac{2\mu(A_\infty)\gamma}{[\mu(A_\infty)^2+y^2]}- \Big(1-\int_T^l\frac{\beta(a,J_\infty,A_\infty)}{V(a)}e^{-\mu(A_\infty)\int_0^a\frac{d\tau}{V(\tau)}}\cos y\int_0^a\frac{d\tau}{V(\tau)}da\Big)\Big)\Big]^2\Big\} -\frac{[\mu(A_\infty)^2+y^2]}{4\mu(A_\infty)e^{-\mu(A_\infty)m}[\delta-\gamma]\sin ym}\times \Big\{1+\Big(1-\int_T^l\frac{\beta(a,J_\infty,A_\infty)}{V(a)}e^{-\mu(A_\infty)\int_0^a\frac{d\tau}{V(\tau)}}\cos y\int_0^a\frac{d\tau}{V(\tau)}da\Big)^2\Big\}>0. \tag{22}$$

Now, we can obtain the result from inequality (22) by first assuming that $\delta-\gamma$ is large and positive and $\sin ym$ is negative, therefore, the first bracketed term is a large negative number whereas the second bracketed term is a small positive number, hence inequality (22) can not be satisfied, and therefore, a nontrivial steady state is locally asymptotically stable. On the other hand, we may assume that $\delta-\gamma$ is large and negative and $\sin ym$ is positive therefore, as before, the first bracketed term is a large negative number whereas the second bracketed term is a small positive number, hence inequality (22) can not be satisfied, and therefore, a nontrivial steady state is locally asymptotically stable. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $\frac{1}{D}=0$, where $\frac{1}{D}$ is given by

$$\frac{1}{D}=[\chi(\mu(A_\infty)+\chi)\gamma+e^{-\mu(A_\infty)m}(\delta-\gamma)(\mu(A_\infty)\chi-y^2)]. \tag{23}$$

Theorem 2.10 Suppose that, $\mu(A_\infty)\geq\chi\geq 0$, $\frac{1}{D}=0$, and, $\Big[e^{\mu(A_\infty)m}\frac{[\mu(A_\infty)-\chi]}{[\mu(A_\infty)+\chi]}+1\Big]\gamma<\delta<0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From equation (13), we obtain the following condition for crossing:

$$e^{-\mu(A_\infty)m}[\delta-\gamma]\Big\{y^2[\mu(A_\infty)+\chi]\cos ym+y[\mu(A_\infty)\chi-y^2]\sin ym\Big\}+$$
$$\gamma\Big\{\mu(A_\infty)\chi^2+\mu(A_\infty)y^2+[\chi y^2-\mu(A_\infty)\chi^2]\cos ym+\chi y[\mu(A_\infty)+\chi]\sin ym\Big\}>0.$$

Now, if we use equation (23), we obtain the following condition for crossing:

$$\begin{aligned}&\delta e^{-\mu(A_\infty)m}y^2[\mu(A_\infty)+\chi]\cos ym+\gamma\mu(A_\infty)y^2[1-e^{-\mu(A_\infty)m}\cos ym]\\&+\gamma\chi y^2[1-e^{-\mu(A_\infty)m}]\cos ym+\gamma\mu(A_\infty)\chi^2[1-\cos ym]>0.\end{aligned}\tag{24}$$

From inequality (24), if we assume that $\cos ym\geq 0$, then we obtain a contradiction. Accordingly, we assume that $\cos ym<0$, and use the assumption that $\Big[e^{\mu(A_\infty)m}\frac{[\mu(A_\infty)-\chi]}{[\mu(A_\infty)+\chi]}+1\Big]\gamma<\delta<0$, we obtain

$$\begin{aligned}&\delta e^{-\mu(A_\infty)m}y^2[\mu(A_\infty)+\chi]\cos ym+\gamma\mu(A_\infty)y^2[1-e^{-\mu(A_\infty)m}\cos ym]\\&+\gamma\chi y^2[1-e^{-\mu(A_\infty)m}]\cos ym+\gamma\mu(A_\infty)\chi^2[1-\cos ym]<\\&\gamma\Big[\frac{\mu(A_\infty)-\chi}{\mu(A_\infty)+\chi}+e^{-\mu(A_\infty)m}\Big]y^2[\mu(A_\infty)+\chi]\cos ym+\gamma\mu(A_\infty)y^2[1-e^{-\mu(A_\infty)m}\cos ym]\\&+\gamma\chi y^2[1-e^{-\mu(A_\infty)m}]\cos ym+\gamma\mu(A_\infty)\chi^2[1-\cos ym]=\\&\gamma\mu(A_\infty)y^2[1+\cos ym]+\gamma\mu(A_\infty)\chi^2[1-\cos ym]<0.\end{aligned}\tag{25}$$

Now, from (25), we see that crossing is impossible, and the proof of the theorem is completed by using Theorem 2.1. This completes the proof of the theorem.

In the next result, we generalize Theorem 2.10 in the sense that we relax the assumption that $\gamma<0$. We note that in such case we need to assume the following in order that crossing of the imaginary axis takes place when $y\neq 0$:

$$\begin{aligned}&\delta A_\infty+\gamma J_\infty+\frac{\chi e^{\mu(A_\infty)m}}{P_\infty}\Big\{\gamma\Big[J_\infty\int_T^l\int_0^a\frac{p_\infty(a)}{V(\sigma)}d\sigma da-A_\infty\int_0^T\int_0^a\frac{p_\infty(a)}{V(\sigma)}d\sigma da\Big]\\&-A_\infty\int_T^l\int_0^a\beta(a,J_\infty,A_\infty)\frac{p_\infty(a)}{V(\sigma)}d\sigma da\Big\}<0.\end{aligned}\tag{26}$$

Theorem 2.11 Suppose that, $\delta<0,\chi\geq 0$, and, $\frac{1}{D}=0$. Then a nontrivial steady state is locally asymptotically stable in each of the following cases:

1)

$$N\cos ym+\gamma\mu(A_\infty)[\chi^2+y^2]\leq 0,\tag{27}$$

when $\delta\neq\gamma$, and condition (26) holds, where y^2 and N are given by

$$y^2=\mu(A_\infty)\chi+\frac{\gamma\chi(\mu(A_\infty)+\chi)}{e^{-\mu(A_\infty)m}(\delta-\gamma)},\tag{28}$$
$$N=e^{-\mu(A_\infty)m}(\delta-\gamma)(\mu(A_\infty)+\chi)y^2+\gamma\chi(y^2-\mu(A_\infty)\chi).\tag{29}$$

2) $\delta = \gamma$.

Proof. To prove 1, we suppose that $\delta \neq \gamma$, then from $\frac{1}{D} = 0$, we obtain equation (28). Also from equation (13), it is easy to see that inequality (27) is the condition for a nontrivial steady state to be locally asymptotically stable provided that condition (26) holds. This proves 1.

To prove 2, we note that if $\delta = \gamma$, then from $\frac{1}{D} = 0$, we obtain that, $\chi = 0$, and, in this case, we find that inequality (27) is satisfied. We also note that, in this case, condition (26) is automatically satisfied. This completes the proof of the theorem.

In order to facilitate our writing, we define L by

$$L = e^{-\mu(A_\infty)m}(\delta - \gamma)(\mu(A_\infty) + \chi) + \chi\gamma. \tag{30}$$

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L = \frac{1}{D} = 0$, and, $\delta < 0$.

Theorem 2.12 Suppose that, $\delta < 0, L = \frac{1}{D} = 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We start by supposing that $\mu(A_\infty) + \chi = 0$, then from $L = 0$, we obtain that $\gamma = 0$, and from $\frac{1}{D} = 0$, we obtain $e^{-\mu(A_\infty)m}\delta\left[\mu(A_\infty)^2 + y^2\right] = 0$, which is impossible since $\delta < 0$, and $\mu(A_\infty) > 0$. Accordingly, $\mu(A_\infty) + \chi \neq 0$, and we can divide in the equation for $L = 0$ to obtain

$$e^{-\mu(A_\infty)m}(\delta - \gamma) = -\frac{\gamma\chi}{[\mu(A_\infty) + \chi]}. \tag{31}$$

Now, we can use equation (31) in the equation for $\frac{1}{D} = 0$, to obtain

$$\frac{\chi\gamma}{[\mu(A_\infty) + \chi]}\left[(\mu(A_\infty) + \chi)^2 - \mu(A_\infty)\chi + y^2\right] = 0, \tag{32}$$

which implies that either $\chi = 0$, or $\gamma = 0$, or $(\mu(A_\infty) + \chi)^2 - \mu(A_\infty)\chi + y^2$. Suppose that $\chi = 0$, then by using $L = 0$, we obtain that $\gamma = \delta$, and hence, we obtain the result from Theorem 2.6. If we suppose that $\gamma = 0$, then from $L = 0$, we obtain that $\delta = 0$, which is impossible. Also, it is easy to see that $(\mu(A_\infty) + \chi)^2 - \mu(A_\infty)\chi + y^2 \neq 0$. Accordingly, only $\chi = 0$ is possible in equation (32). This completes the proof of the theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L = N = 0$, and $\delta < 0$.

Theorem 2.13 Suppose that, $\delta < 0, L = N = 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that in this case, from $N = Ly^2 - \gamma\mu(A_\infty)\chi^2$, we obtain that $\chi^2\gamma = 0$, which implies that either $\chi = 0$, or $\gamma = 0$. If $\chi = 0$, then from $L = 0$, we obtain $\delta = \gamma$, accordingly, local asymptotic stability follows from Theorem 2.6. If $\gamma = 0$, then from $L = N = 0$, and using L'Hôpital's rule, we obtain

$$y^2 = \frac{\mu(A_\infty)\chi^2\gamma}{\left[e^{-\mu(A_\infty)m}(\delta-\gamma)(\mu(A_\infty)+\chi)+\chi\gamma\right]} = -\mu(A_\infty)^2 < 0, \tag{33}$$

which is impossible. This completes the proof of the theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $L = 0$, and $\delta = \gamma < 0$.

Theorem 2.14 Suppose that, $L = 0$, and, $\delta = \gamma < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From $L = 0$, we obtain that $\chi = 0$, and hence the result follows from Theorem 2.6. This completes the proof of the theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $\frac{1}{D} = N = 0$, and, $\delta = \gamma < 0$.

Theorem 2.15 Suppose that, $\frac{1}{D} = N = 0$, and, $\delta = \gamma < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From $N = 0$, we obtain $\chi\gamma[y^2 - \chi\mu(A_\infty)] = 0$. Therefore, either $\chi = 0$, and hence the result follows from Theorem 2.6, or $y^2 = \chi\mu(A_\infty)$, and in this case, by using $\frac{1}{D} = 0$, we obtain $\chi(\mu(A_\infty) + \chi)\gamma = 0$. We note that, in this case, if we assume that $\mu(A_\infty) + \chi = 0$, then we obtain that $y^2 = -\mu(A_\infty)^2$, which is impossible. This completes the proof of the theorem.

In the next result, we prove that a nontrivial steady state is locally asymptotically stable when, $\frac{1}{D} = N = 0$, and $\delta < 0, \delta A_\infty + \gamma J_\infty < 0$.

Theorem 2.16 Suppose that, $\frac{1}{D} = N = 0, \delta < 0$, and, $\delta A_\infty + \gamma J_\infty < 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. We note that when $\frac{1}{D} = L = 0$, then by Theorem 2.12, we obtain the result.

Therefore, we only need to consider the case $L \neq 0$. Also if $\delta = \gamma$, then we obtain the result via Theorem 2.15. Accordingly, we obtain the following two equations for y^2 :

$$y^2 = \frac{\mu(A_\infty)\chi^2}{L}\gamma, \tag{34}$$

$$y^2 = \mu(A_\infty)\chi + \frac{\chi(\chi+\mu(A_\infty))}{e^{-\mu(A_\infty)m}(\delta-\gamma)}\gamma. \tag{35}$$

Now, using equations (34)-(35), we obtain

$$\chi(\mu(A_\infty)+\chi)\left\{\left[e^{-\mu(A_\infty)m}(\delta-\gamma)\right]^2 + \frac{L}{\mu(A_\infty)}\gamma\right\} = 0. \tag{36}$$

From equation (36), we obtain that either $\chi = 0$ or $\mu(A_\infty) + \chi = 0$, or $\left[e^{-\mu(A_\infty)m}(\delta - \gamma)\right]^2 + \frac{L}{\mu(A_\infty)}\gamma = 0$. If $\mu(A_\infty) + \chi = 0$, then from equation (35), we obtain $y^2 = -\mu(A_\infty)^2$, which is impossible. Also, if $\left[e^{-\mu(A_\infty)m}(\delta - \gamma)\right]^2 + \frac{L}{\mu(A_\infty)}\gamma = 0$, then $\left[e^{-\mu(A_\infty)m}(\delta - \gamma)\right]^2 = -\frac{L}{\mu(A_\infty)}\gamma$. But this is also impossible since from equation (34), we obtain $\frac{L}{\mu(A_\infty)}\gamma > 0$, and therefore, $\left[e^{-\mu(A_\infty)m}(\delta - \gamma)\right]^2 + \frac{L}{\mu(A_\infty)}\gamma \neq 0$. Hence $\chi = 0$ is the only solution of equation (36). But in this case by the assumptions $\delta < 0, \delta A_\infty + \gamma J_\infty < 0$, crossing with $y = 0$ is not possible by Theorem 3.3 in El-Doma (to appear), and the result is obtained by using Theorem 2.1. This completes the proof of the theorem.

In the following result, we describe the stability of a nontrivial steady state when, $\delta = \gamma < 0, N = 0$, and, $\chi \leq 0$.

Theorem 2.17 Suppose that $\delta = \gamma < 0, N = 0$, and, $\chi \leq 0$. Then a nontrivial steady state is locally asymptotically stable.

Proof. From N=0, we obtain $\gamma\chi(y^2 - \mu(A_\infty)\chi) = 0$, which implies that $\chi = 0$, since $\chi \leq 0$. Accordingly, the result follows from Theorem 2.6. This completes the proof of the theorem.

In the following result, we use inequality (26) to obtain the result of Theorem 2.9.

Theorem 2.18 Suppose that, $\delta < 0, \delta \neq \gamma, \chi = 0, \delta A_\infty + \gamma J_\infty < 0$, and, $\sin ym \neq 0$. Then the result of Theorem 2.9 holds.

Proof. We only need to observe that in this case inequality (26) becomes $\delta A_\infty + \gamma J_\infty < 0$. The remaining steps for the proof are the same as in the proof of Theorem 2.9. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $N = 0$, where N is given by equation (29).

Theorem 2.19 Suppose that, $N = 0, \frac{1}{D} \neq 0, \sin ym \neq 0$, and, $\delta < 0$. Then a nontrivial steady state is locally asymptotically stable if $L = 0$. If $L \neq 0$, then a nontrivial steady state is locally asymptotically stable if inequality (26) holds, $\chi \geq 0$, and, the following inequality holds:

$$1 + \left\{1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da - \frac{\mu(A_\infty)\gamma}{\Delta_0}[y^2 + \chi^2]\right\} \times \left\{\frac{D\gamma[y^2 + \chi^2] - \cos ym}{\sin ym}\right\} \leq 0, \tag{37}$$

where y^2 is given by

$$y^2 = \frac{\gamma\mu(A_\infty)\chi^2}{\left[(\mu(A_\infty) + \chi)e^{-\mu(A_\infty)m}(\delta - \gamma) + \chi\gamma\right]}. \tag{38}$$

Proof. Suppose that $L = 0$, then the result follows from Theorem 2.13.

Now, suppose that $L \neq 0$, then we can solve for y^2 from equation (29) to obtain equation (38). Also, from equation (13), we obtain

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da - \frac{\gamma}{\Delta_0}\mu(A_\infty)(\chi^2 + y^2) = \frac{y \sin ym}{\Delta_0 D}. \quad (39)$$

From equation (39), we can solve for y, and then use equation (14) to obtain inequality (37). This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial steady state when, $N \neq 0, \frac{1}{D} \neq 0, \sin ym \neq 0$, and, condition (26) holds.

Theorem 2.20 Suppose that, $\delta < 0, \chi \geq 0, N \neq 0, \frac{1}{D} \neq 0, \sin ym \neq 0$, and, condition(26) holds. Then a nontrivial steady state is locally asymptotically stable if

$$\frac{N}{\Delta_0 \sin ym}\Big\{\Big[\sin ym + \frac{\Delta_0}{2N}\Big]^2 + \Big[\cos ym + \frac{W}{2N}\Big]^2\Big\} - \frac{1}{4N\Delta_0 \sin ym}\Big\{\Delta_0^2 + \hat{W}^2\Big\} \leq 0, \quad (40)$$

where $D, N,$ are given, respectively, by (23), (29), and $W, \hat{W}$ are defined as follows

$$W = \gamma[y^2 + \chi^2][\mu(A_\infty) - DN] - \Delta_0\Big[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da\Big], \quad (41)$$

$$\hat{W} = \gamma[y^2 + \chi^2][\mu(A_\infty) + DN] - \Delta_0\Big[1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da\Big]. \quad (42)$$

Proof. From equation (13), we obtain

$$1 - \int_T^l \frac{\beta(a, J_\infty, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} \cos y \int_0^a \frac{d\tau}{V(\tau)} da - \frac{N}{\Delta_0}\cos ym - \frac{\gamma\mu(A_\infty)}{\Delta_0}[y^2 + \chi^2] = \frac{y \sin ym}{\Delta_0 D}. \quad (43)$$

From $\frac{1}{D} \neq 0$, and $\sin ym \neq 0$, we can solve for y in equation (43), and then use equation (14) to obtain (40). This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial trivial steady state when, $\sin ym = 0$. We note that this result is similar to Theorem 2.11, and therefore, proof is omitted.

Theorem 2.21 Suppose that, $\delta < 0, \chi \geq 0$, condition (26) holds, and, $\sin ym = 0$. Then a nontrivial steady state is locally asymptotically stable if

$$N \cos ym + \gamma\mu(A_\infty)[\chi^2 + y^2] \leq 0, \quad (44)$$

where $y = \frac{n\pi}{m}, n = \pm 1, \pm 2, \pm 3, ...$

In the next result, we describe the stability of a nontrivial steady state when, $\sin ym = 0 = L$, where L is given by equation (30).

Theorem 2.22 Suppose that, $\delta < 0, \chi \geq 0, L = 0$, and, $\sin ym = 0$. Then a nontrivial steady state is locally asymptotically stable in each of the following cases:

1) $\gamma < 0$,
2) $e^{-\mu(A_\infty)m} - \dfrac{\chi}{[\mu(A_\infty) + \chi]} > 0$.

Proof. To prove 1, we note that from equation (27), and $L = 0$, we obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\gamma\mu(A_\infty)\Big[y^2 + \chi^2(1 - \cos ym)\Big] \leq 0. \tag{45}$$

Accordingly, since by 1, $\gamma < 0$, the result follows from inequality (45) since inequality (26) is automatically satisfied. This proves 1.

To prove 2, we note that since $\mu(A_\infty) + \chi > 0$, then from $L = 0$, we obtain

$$\delta e^{-\mu(A_\infty)m} = \gamma\Big[e^{-\mu(A_\infty)m} - \frac{\chi}{[\mu(A_\infty) + \chi]}\Big]. \tag{46}$$

Now, using 2, the result follows easily from equation (46) since we have, $\gamma < 0$, accordingly, we see that (45) is satisfied. This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial trivial steady state when, $\sin ym = N = 0$.

Theorem 2.23 Suppose that, $\delta < 0, \chi \geq 0, \sin ym = N = 0$, and, condition (26) holds. Then a nontrivial steady state is locally asymptotically stable if $\dfrac{\chi[\mu(A_\infty)\chi - (\frac{n\pi}{m})^2]}{[\mu(A_\infty) + \chi]} + (\dfrac{n\pi}{m})^2 e^{-\mu(A_\infty)m} > 0, n = \underline{+}1, \underline{+}2, \underline{+}3, ...$

Proof. We start by noting that if $L = 0$, then by Theorem 2.13, we obtain the result.

Accordingly, we assume that $L \neq 0$, and therefore, $y^2 = \dfrac{\gamma\mu(A_\infty)\chi^2}{L} = (\dfrac{n\pi}{m})^2$, hence, from $N = 0$, we obtain

$$\gamma\Big[\mu(A_\infty)\chi^2 + e^{-\mu(A_\infty)m}(\mu(A_\infty) + \chi)(\frac{n\pi}{m})^2 - (\frac{n\pi}{m})^2\chi\Big] = \delta(\frac{n\pi}{m})^2 e^{-\mu(A_\infty)m}(\mu(A_\infty) + \chi).$$

So, since $\mu(A_\infty) + \chi \neq 0$, then we obtain the result from $\dfrac{\chi[\mu(A_\infty)\chi - (\frac{n\pi}{m})^2]}{[\mu(A_\infty) + \chi]} + (\dfrac{n\pi}{m})^2 e^{-\mu(A_\infty)m} > 0, n = \underline{+}1, \underline{+}2, \underline{+}3, ...$ This completes the proof of the theorem.

In the next result, we describe the stability of a nontrivial trivial steady state when, $\mu(A_\infty) \geq \chi \geq 0 = \sin ym$, and $\gamma \leq \delta < 0$.

Theorem 2.24 Suppose that, $\mu(A_\infty) \geq \chi \geq 0 = \sin ym$, and, $\gamma \leq \delta < 0$. Then a nontrivial trivial steady state is locally asymptotically stable.

Proof. We note that in this case by using $\sin ym = 0$, we obtain condition (24) for crossing.

From (24), it is easy to see that if $\cos ym = 1$, then we obtain a contradiction.

Accordingly, we assume that $\cos ym = -1$, to obtain the following condition for crossing:

$$y^2 e^{-\mu(A_\infty)m}(\gamma - \delta)(\mu(A_\infty) + \chi) + \gamma y^2(\mu(A_\infty) - \chi) + 2\gamma\mu(A_\infty)\chi^2 > 0, \tag{47}$$

which is impossible, and the result is completed by using Theorem 2.1. This completes the proof of the theorem.

Example 1: In this example, we consider the case when $\beta(a, J, A) = \beta_0, \mu(a, J, A) = \mu(A), V(a, J, A) = V(a)$, where β_0 is a constant.

In this case from equation (6) in El-Doma (to appear), we obtain

$$\mu(A_\infty)e^{\mu(A_\infty)\int_0^T \frac{d\tau}{V(\tau)}} = \beta_0. \tag{48}$$

Also, from equation (9) in El-Doma (to appear), we obtain

$$P_\infty = A_\infty e^{\mu(A_\infty)\int_0^T \frac{d\tau}{V(\tau)}}. \tag{49}$$

Now, if we can solve for A_∞ from equation (48), then P_∞ is determined from equation (49), and accordingly, J_∞ is given by

$$J_\infty = P_\infty\Big[1 - e^{-\mu(A_\infty)\int_0^T \frac{d\tau}{V(\tau)}}\Big]. \tag{50}$$

Therefore, using Theorem 2.2, we obtain that a nontrivial steady state is locally asymptotically stable if $\chi > 0$, and is unstable if $\chi < 0$.

Example 2: In this example, we consider the case when $\beta(a, J, A) = \beta_0(a)e^{-c_1 P}$, $\mu(a, J, A) = \mu(A), V(a, J, A) = V(a)$, where c_1 is a constant.

In this case from equation (6) in El-Doma (to appear), we obtain

$$1 = e^{-c_1 P_\infty}\int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} da. \tag{51}$$

Also, from equation (9) in El-Doma (to appear), we obtain

$$P_\infty = A_\infty e^{\mu(A_\infty)\int_0^T \frac{d\tau}{V(\tau)}}. \tag{52}$$

Now, using equation (52) in equation (51), we can see that if

$$\int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(0))\int_0^a \frac{d\tau}{V(\tau)}} da > 1, \tag{53}$$

and

$$\int_T^\infty \frac{\beta_0(a)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} da < +\infty, \tag{54}$$

then a nontrivial steady state exists.

Using equations (52), (4), we obtain

$$\chi = A_\infty \mu'(A_\infty). \tag{55}$$

Accordingly, using Theorem 2.6, we obtain that a nontrivial steady state is locally asymptotically stable in each of the following cases:

1) $\mu(A) = c_2 A$, where c_2 is a positive constant,
2) $\mu(A) = c_2\sqrt{A}$.

Example 3: In this example, we consider the case when $\beta(a, J, A) = \beta(a, A), \mu(a, J, A) = \mu(A), V(a, J, A) = V(a)$. We note that in this case adults control the population in terms of birth and death. For the case when juveniles control the population in terms of birth, see El-Doma (to appear) and Cushing, et al. (1991).

We also note that in this case, from inequality (1), we obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\int_T^l \frac{e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}}}{V(a)}\Big|\Big[\beta(a, A_\infty) + \int_T^l \beta_A(a, A_\infty)p_\infty(a)da\Big]\Big|da$$
$$+\int_T^l\int_0^a F(a,\sigma)\Big|g_A(\sigma, A_\infty)\Big|d\sigma da + \tag{56}$$
$$\int_T^l\int_T^l\int_0^a \frac{e^{-\mu(A_\infty)\int_0^b \frac{d\tau}{V(\tau)}}}{V(b)}F(a,\sigma)\Big|g_A(\sigma, A_\infty)\Big[\beta(b, A_\infty) - \beta(a, A_\infty)\Big]\Big|d\sigma da db < 1.$$

From Theorem 3.2 in El-Doma (to appear), we obtain the following condition for the instability of a nontrivial steady state:

$$R_A(J_\infty, A_\infty) > 0.$$

Also, in this case from equation (6) in El-Doma (to appear), we obtain

$$1 = \int_T^\infty \frac{\beta(a, A_\infty)}{V(a)} e^{-\mu(A_\infty)\int_0^a \frac{d\tau}{V(\tau)}} da. \tag{57}$$

Note that P_∞ satisfies equation (52). So, the positive solutions of equation (57) determine the nontrivial steady states.

If we assume that $\beta(a, A) = \dfrac{\beta_0}{A^n}$, where β_0 is a constant, $n = 1, 2, ...$; and $\mu(A) = \mu_0 = constant$. Then from inequality (56), we obtain the following condition for the local asymptotic stability of a nontrivial steady state:

$$\frac{\beta_0(n-1)e^{-\mu_0\int_0^T \frac{d\tau}{V(\tau)}}}{\mu_0 A_\infty^n} < 1. \tag{58}$$

We note that inequality (58) is automatically satisfied when, $n = 1$.

Also, note that by using equation (57), we obtain that (58) becomes

$$n < 2,$$

which means that $n = 1$.

Regarding Example 1 - Example 3, we note that we can use Theorem 2.3 in El-Doma (to appear), and Corollary 3.10 in El-Doma (to appear) to show that these steady states as well as their stability results remain unchanged if each of the vital rates is multiplied by any positive function $f(J, A) \in C^1(\mathbb{R}^{+2})$.

3. Conclusion

In this paper, we continued our study of size-structured population dynamics models, started in El-Doma (to appear). The present study assumed that the population is divided into adults and juveniles, the death rate and the growth rate assumed special cases such that the former depends on adults only and the latter depends on size only, and the maximum size for an individual in the population is infinite.

In our assumption that the death rate depends on adults only, we are motivated by the fact that many species protect their young (juveniles) by sheltering and caring, though this is species specific. Also when disturbed or attacked by predators, for example, some females even take their young into their mouth, for example, see Taborsky (2006).

In our study of the local asymptotic stability of a nontrivial steady state, we identified three demographic parameters: $\delta, \gamma, \chi,$ where $\delta,$ is given by equation (22) in El-Doma (to appear), which represents the total change in the birth rate, at the steady state, due to changes in adults only; $\gamma,$ is given by equation (23) in El-Doma (to appear), which represents the total change in the birth rate, at the steady state, due to changes in juveniles only; and $\chi,$ is given by equation (4), which represents the total change in the death rate, at the steady state, due to changes in adults only.

We obtained several conditions, depending on the three mentioned demographic parameters, for the (in)stability of the nontrivial steady states. We also determined relations that lead to similar conditions on the three demographic parameters, for example, see Theorem 2.12 - Theorem 2.16. We also illustrated our stability results by several examples.

We note that our model in this paper generalized that given in El-Doma (2008 a), where juveniles are not considered. We retained all the related stability results given therein. We also note that this case linked our study of the stability of our size-structured population dynamics model to the study of the classical Gurtin-MacCamy's age-structured population dynamics model given in Gurtin, et al. (1974), specifically, the studies for the stability given in Gurney, et al. (1980) and Weinstock, et al. (1987), in fact, the characteristic equation for this special case, when juveniles are not considered i.e. when, $T = 0,$ has the same qualitative properties as the characteristic equation of the Gurtin-MacCamy's age-structured population dynamics model, for example, see El-Doma (2008).

We also note that in our first paper in this series of three papers, we studied the general model with general vital rates, and determined the steady states and obtained general conditions for the (in)stability of the (non)trivial steady states as well as several special cases.

In the last paper of our series, further stability results will be given for the case when, $V(a, J, A) = V(a), \mu(a, J, A) = \mu(J),$ and also the case when, $V(a, J, A) = V(a), \mu(a, J, A) = \mu(a).$

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