



## ANALYSIS OF AN SIRS AGE-STRUCTURED EPIDEMIC MODEL WITH VACCINATION AND VERTICAL TRANSMISSION OF DISEASE

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### Abstract

An SIRS age-structured epidemic model for a vertically as well as horizontally transmitted disease under vaccination is investigated when the fertility, mortality and removal rates depend on age and the force of infection of proportionate mixing assumption type, and vaccination wanes over time. We prove the existence and uniqueness of solution to the model equations, and show that solutions of the model equations depend continuously on the initial age-distributions. Furthermore, we determine the steady states and obtain an explicitly computable threshold condition, in terms of the demographic and epidemiological parameters of the model; we then study the stability of the steady states. We also compare the behavior of the model with the one without vertical transmission.

**Keywords:** Vertical transmission; Horizontal transmission; Age-structure; Epidemic; Stability; Proportionate mixing

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### 1. Introduction

In this paper, we study an SIRS age-structured epidemic model, where age is assumed to be the chronological age i.e. the time since birth. The disease causes so few fatalities that they can be neglected, and is horizontally as well as vertically transmitted. Horizontal transmission is the passing of infection through some direct or indirect contact with infected individuals, for example, malaria and tuberculosis are horizontally transmitted diseases. Vertical transmission is the passing of infection from parents to newborn or unborn offspring, for example, AIDS, Chagas and Hepatitis B are vertically (as well as horizontally) transmitted diseases. Vertical transmission plays an important role in maintaining some diseases, for example, see Busenberg, et al. (1993), (1988, p. 1379), (1982), (1988, p. 181), (1990) and Busenberg (1986). In Fine (1975), several examples of vertically transmitted diseases are given, and in Busenberg, et al. (1993), a book is devoted for the study of the models and dynamics of vertically transmitted diseases.

Several recent papers have dealt with age-dependent vaccination models, for example, Hethcote (1983), (1989), (1997), (2000), Dietz, et al. (1985), Dietz (1981), Katzmann, et al. (1984), Schenzle (1984), Anderson, et al. (1999), El-Doma (2000), (2001), (2005), (2006), Müller (1994), (1998), Haderler, et al. (1996), Knox (1980), McLean (1986), Coutinho, et al.

(1993), Lopez, et al. (2000), Greenhalgh (1988),(1990), Castillo-Chavez, et al. (1998), Li, et al. (2004), and Thieme (2001). Much of this previous work assumed that the total population has a fixed size. As one might expect there are situations, for example, in developing countries, where it is necessary to consider total population sizes that vary with time, mortality due to disease and reduction of birth rate due to infection. Some of the effects of such assumptions are considered by McLean (1986), El-Doma (2001), (2004) and Haderler, et al. (1996).

We note that several papers have dealt with SIR age-structured epidemic models, but without vaccination, for example, Thieme (1990), Andreasen (1995), Chipot, et al. (1995), Inaba (1990), Cha, et al. (1998), Greenhalgh (1987) and Castillo-Chavez, et al. (1989). Also, we note that some vaccines wane over time giving rise to SIRS type models, for example, see, Li, et al. (2004).

In this paper, we study an SIRS age-structured epidemic model when the disease is vertically as well as horizontally transmitted, and therefore our results generalize those in Li, et al. (2004), where vertical transmission is not considered. We prove the existence and uniqueness of solution to the model equations and show that solutions of the model equations depend continuously on the initial age-distributions. We determine the steady states of the model by proving a threshold theorem and obtaining an explicitly computable threshold  $R_v$ , in terms of the demographic and epidemiological parameters of the model, known as the reproduction number in the presence of vaccination strategy  $v(a)$ , as in Haderler, et al. (1996) or the net replacement ratio, as in Thieme (2001). And hence, we show that  $R_v$  increases with  $q$ , which is the probability of vertically transmitting the disease (see section 2 for definitions) and therefore, increases the likelihood that an endemic will occur; also  $R_v$  is used to determine a critical vaccination coverage which will eradicate the disease with minimum vaccination coverage. In addition, we show that the model gives rise to a continuum of positive endemic equilibriums in the case of non-fertile infectibles. This situation does not occur if there is no vertical transmission, for example, see, Li, et al. (2004). We also study the stability of the steady states.

We study the stability of the disease-free equilibrium and show that the disease-free equilibrium is locally asymptotically stable if  $R_v < 1$ , and unstable if  $R_v > 1$ . Also, we show that the disease-free equilibrium is globally stable if  $R_0 < 1$ , where  $R_0$  is the basic reproduction number, and is interpreted as the average number of secondary infections that occur when an infective is introduced into a totally susceptible population. For endemic equilibriums, we obtain a complicated characteristic equation, that allows us to prove local asymptotic stability, in some special cases. In addition, we derive general formulas for the characteristic equations, in terms of integral equations. Although we do not obtain explicit formulas for the solutions of these integral equations, we will use these integral equations to deduce that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

The organization of this paper is as follows: in section 2 we describe the model and obtain the model equations; in section 3 we reduce the model equations to several subsystems and prove the existence and uniqueness of solution as well as the continuous dependence on initial age-distributions; in section 4 we determine the steady states; in section 5 we study the stability of the steady states; in section 6 we conclude our results.

## 2. The Model

We consider an age-structured population of variable size exposed to a communicable disease. The disease is vertically as well as horizontally transmitted and causes so few fatalities that they can be neglected. We assume the following.

1.  $s(a,t)$ ,  $i(a,t)$  and  $r(a,t)$ , respectively, denote the age-density for susceptible, infective and immune individuals of age  $a$  at time  $t$ . Then

$$\int_{a_1}^{a_2} s(a,t) da = \text{total number of susceptible individuals at time } t \text{ of ages between } a_1 \text{ and } a_2,$$

$$\int_{a_1}^{a_2} i(a,t) da = \text{total number of infective individuals at time } t \text{ of ages between } a_1 \text{ and } a_2.$$

And similarly for  $r(a,t)$ . We assume that the total population consists entirely of susceptible, infective and immune individuals.

2. Let  $k(a,a')$  denote the probability that a susceptible individual of age  $a$  is infected by an infective of age  $a'$ . We further assume that,  $k(a,a') = k_1(a)k_2(a')$ , which is known as the ‘‘proportionate mixing assumption’’, see Dietz, et al. (1985). Therefore, the horizontal transmission of the disease for the susceptible individuals occurs at the following rate:

$$k_1(a)s(a,t) \int_0^\infty k_2(a')i(a',t) da',$$

where  $k_1(a)$  and  $k_2(a)$  are bounded, nonnegative, continuous functions of  $a$ . The term

$k_1(a) \int_0^\infty k_2(a')i(a',t) da$ , is called ‘‘force of infection’’ and we let

$$\lambda(t) = \int_0^\infty k_2(a')i(a',t) da.$$

And immune individuals are infected due to waning of vaccine over time at the following rate:  $\varepsilon k_1(a)r(a,t)\lambda(t)$ , where  $\varepsilon$  is a positive real number  $\in [0,1]$ .

3. The fertility rate  $\beta(a)$  is a nonnegative, continuous function, with compact support  $[0, A]$ , ( $A > 0$ ). The number of births of susceptible individuals per unit time is given by

$$s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t) + r(a,t)] da, q \in [0,1],$$

where  $q$  is the probability of vertically transmitting the disease. Accordingly all newborns from susceptible and immune individuals are susceptible, but a portion  $q$  of newborns from infected parents are infective, i.e., they acquire the disease via birth (vertical transmission) and therefore,  $i(0,t) = q \int_0^\infty \beta(a)i(a,t) da$ , and  $r(0,t) = 0$ .

4. The death rate  $\mu(a)$  is the same for susceptible, infective and immune individuals, and  $\mu(a)$  is a nonnegative, continuous function and  $\exists a_0 \in [0, \infty)$  such that  $\mu(a) > \bar{\mu} > 0 \forall a > a_0$  and  $\mu(a_2) > \mu(a_1) \forall a_2 > a_1 > a_0$ .
5. The cure rate  $\gamma(a)$  is a bounded, nonnegative, continuous function of  $a$ .
6. The vaccination rate  $\nu(a)$  is a bounded, nonnegative, continuous function of  $a$ .
7. The vaccination wanes in immune individuals and they become susceptible at a rate  $\delta(a)$  which is a bounded, nonnegative, continuous function of  $a$ .
8. The initial age distributions  $s(a,0) = s_0(a)$ ,  $i(a,0) = i_0(a)$ , and  $r(a,0) = r_0(a)$  are continuous, nonnegative and integrable functions of  $a \in [0, \infty)$ .

These assumptions lead to the following system of nonlinear integro-partial differential equations with non-local boundary conditions, which describes the dynamics of the transmission of the disease.

$$\left\{ \begin{array}{l} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + [\mu(a) + \nu(a)]s(a,t) = -k_1(a)s(a,t)\lambda(t) + \delta(a)r(a,t), a > 0, t > 0, \\ \frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} + [\mu(a) + \gamma(a)]i(a,t) = k_1(a)s(a,t)\lambda(t) + \varepsilon k_1(a)r(a,t)\lambda(t), a > 0, t > 0, \\ \frac{\partial r(a,t)}{\partial a} + \frac{\partial r(a,t)}{\partial t} + [\mu(a) + \delta(a)]r(a,t) \\ \quad = \nu(a)s(a,t) + \gamma(a)i(a,t) - \varepsilon k_1(a)r(a,t)\lambda(t), a > 0, t > 0, \\ s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t) + r(a,t)]da, t \geq 0, \\ i(0,t) = q \int_0^\infty \beta(a)i(a,t)da, t \geq 0, \\ r(0,t) = 0, t \geq 0, \\ \lambda(t) = \int_0^\infty k_2(a)i(a,t)da, t \geq 0, \\ s(a,0) = s_0(a), i(a,0) = i_0(a), r(a,0) = r_0(a), a \geq 0. \end{array} \right. \quad (2.1)$$

We note that problem (2.1) is an SIRS epidemic model, the same model but with  $q = 0$ , i.e., the case of no vertical transmission, is dealt with in Li, et al. (2004), and the steady states are determined and the stability of the disease-free equilibrium is studied.

In what follows, we show that problem (2.1) has a unique solution that exists for all time. Furthermore, we show that solutions of problem (2.1) depend continuously on the initial age-distributions. Also, we determine the steady states and study their stability.

### 3. Reduction of the Model, the Existence and Uniqueness of Solution and Continuous Dependence on Initial Age-Distributions

In this section, we develop some preliminary formal analysis of problem (2.1) and show that problem (2.1) has a unique solution that exists for all time. Furthermore, we show that solutions of problem (2.1) depend continuously on the initial age-distributions.

We define  $p(a,t)$  by  $p(a,t) = s(a,t) + i(a,t) + r(a,t)$ . Then from (2.1), by adding the equations, we find that  $p(a,t)$  satisfies the following McKendrick-Von Forester equation:

$$\left\{ \begin{array}{l} \frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0, a > 0, t > 0, \\ p(0,t) = B(t) = \int_0^\infty \beta(a)p(a,t)da, t \geq 0, \\ p(a,0) = p_0(a) = s_0(a) + i_0(a) + r_0(a), a \geq 0. \end{array} \right. \quad (3.1)$$

Note that problem (3.1) has a unique solution that exists for all time, see Bellman, et al.

(1963), Feller (1941) and Hoppensteadt (1975). The unique solution is given by

$$p(a,t) = \begin{cases} p_0(a-t)\pi(a)/\pi(a-t), & a > t, \\ B(t-a)\pi(a), & a < t, \end{cases} \quad (3.2)$$

where  $\pi(a)$  is given by

$$\pi(a) = e^{-\int_0^a \mu(\tau) d\tau},$$

and  $B(t)$  has the following asymptotic behavior as  $t \rightarrow \infty$ :

$$B(t) = [c + \theta(t)]e^{p^*t}, \quad (3.3)$$

where  $p^*$  is the unique real number which satisfies the following characteristic equation:

$$\int_0^\infty \beta(a)\pi(a)e^{-p^*a} da = 1, \quad (3.4)$$

$\theta(t)$  is a function such that  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  and  $c$  is a constant.

Using (3.1)-(3.2), we obtain that  $B(t)$  satisfies

$$B(t) = \int_0^\infty \beta(a)\pi(a)B(t-a)da + \int_t^\infty \beta(a)p_0(a-t)\frac{\pi(a)}{\pi(a-t)} da. \quad (3.5)$$

Using (3.5) and *Gronwall's* inequality, we obtain

$$B(t) \leq \|\beta(a)\|_\infty \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\beta(a)\|_\infty - \mu_*)t}, \quad (3.6)$$

where  $\mu_*$  is given by

$$\mu_* = \inf_{a \in [0,\infty)} \mu(a). \quad (3.7)$$

From (3.2) and (3.6), we obtain the following a priori estimate:

$$\int_0^\infty p(a,t)da \leq \|p_0(a)\|_{L^1[0,\infty)} e^{(\|\beta(a)\|_\infty - \mu_*)t}. \quad (3.8)$$

Also, from (2.1),  $s(a,t)$ ,  $i(a,t)$  and  $r(a,t)$  satisfy the following systems of equations:

$$\begin{cases} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + [\mu(a) + \nu(a)]s(a,t) = -k_1(a)s(a,t)\lambda(t) + \delta(a)r(a,t), a > 0, t > 0, \\ s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t) + r(a,t)]da, t \geq 0, \\ s(a,0) = s_0(a), a \geq 0, \end{cases} \quad (3.9)$$

$$\begin{cases} \frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} + [\mu(a) + \gamma(a)]i(a,t) = k_1(a)s(a,t)\lambda(t) + \varepsilon k_1(a)r(a,t)\lambda(t), a > 0, t > 0, \\ i(0,t) = q \int_0^\infty \beta(a)i(a,t)da, t \geq 0, \\ i(a,0) = i_0(a), a \geq 0, \end{cases} \quad (3.10)$$

$$r(a,t) = p(a,t) - s(a,t) - i(a,t). \quad (3.11)$$

By integrating problem (3.9) along characteristic lines  $t - a = \text{const}$ , we find that  $s(a, t)$  satisfies

$$s(a, t) = \begin{cases} s_0(a-t)e^{-\int_0^t [\mu(a-t+\tau) + \nu(a-t+\tau) + \delta(a-t+\tau) + k_1(a-t+\tau)\lambda(\tau)] d\tau} \\ + \int_0^t e^{-\int_\sigma^t [\mu(a-t+\tau) + \nu(a-t+\tau) + \delta(a-t+\tau) + k_1(a-t+\tau)\lambda(\tau)] d\tau} \times \\ \delta(a-t+\sigma) [p(a-t+\sigma, \sigma) - i(a-t+\sigma, \sigma)] d\sigma, & a > t \\ [B(t-a) - i(0, t-a)] \pi(a) e^{-\int_0^a [\nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(t-a+\tau)] d\tau} \\ + \int_0^a e^{-\int_\sigma^a [\mu(\tau) + \nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(t-a\tau)] d\tau} \times \\ \delta(\sigma) [p(\sigma, t-a+\sigma) - i(\sigma, t-a+\sigma)] d\sigma, & a < t, \end{cases} \quad (3.12)$$

By integrating problem (3.10) along characteristic lines  $t - a = \text{const}$ , and using (3.12), we find that  $i(a, t)$  satisfies

$$i(a, t) = \begin{cases} i_0(a-t)e^{-\int_0^t [\mu(a-t+\tau) + \gamma(a-t+\tau) + \varepsilon k_1(a-t+\tau)\lambda(\tau)] d\tau} \\ + \int_0^t e^{-\int_\sigma^t [\mu(a-t+\tau) + \gamma(a-t+\tau) + \varepsilon k_1(a-t+\tau)\lambda(\tau)] d\tau} k_1(a-t+\sigma)\lambda(\sigma) \times \\ [(1-\varepsilon)\{s_0(a-t)e^{-\int_0^\sigma [\mu(a-t+\tau) + \nu(a-t+\tau) + \delta(a-t+\tau) + k_1(a-t+\tau)\lambda(\tau)] d\tau} \\ + \int_0^\sigma e^{-\int_b^\sigma [\mu(a-t+\tau) + \nu(a-t+\tau) + \delta(a-t+\tau) + k_1(a-t+\tau)\lambda(\tau)] d\tau} \delta(a-t+b)\} \\ [p(a-t+b, b) - i(a-t+b, b)] db + \varepsilon p(a-t+\sigma, \sigma)] d\sigma, & a > t, \\ i(0, t-a)\pi_2(a) e^{-\varepsilon \int_0^a k_1(\tau)\lambda(t-a+\tau) d\tau} \\ + \int_0^a e^{-\int_\sigma^a [\mu(\tau) + \gamma(\tau) + \varepsilon k_1(\tau)\lambda(t-a+\tau)] d\tau} k_1(\sigma)\lambda(t-a+\sigma) \times \\ [(1-\varepsilon)\{[B(t-a) - i(0, t-a)] e^{-\int_0^\sigma [\mu(\tau) + \nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(t-a+\tau)] d\tau} \\ + \int_0^\sigma e^{-\int_b^\sigma [\mu(\tau) + \nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(t-a\tau)] d\tau} \delta(b)\} \\ [p(b, t-a+b) - i(b, t-a+b)] db + \varepsilon p(\sigma, t-a+\sigma)] d\sigma, & a < t, \end{cases} \quad (3.13)$$

where  $\pi_2(a)$  is given by

$$\pi_2(a) = \pi(a) e^{-\int_0^a \gamma(\tau) d\tau}. \quad (3.14)$$

It is worth noting that if we can establish a solution for problem (3.13), then a solution for problem (3.12) is determined, and consequently a solution for problem (2.1) is determined by using equation (3.11). To establish the existence and uniqueness of solution to problem (2.1), we define the following set  $E$  to satisfy:

$$E = \{i(a, t) : i(\cdot, t) \in L^1([0, \infty)); C[0, t_0], a \in [0, \infty), t \in [0, t_0], \|i(a, t)\| = \sup_{t \in [0, t_0]} \|i(a, t)\|_{L^1}\},$$

where  $C[0, t_0]$  denotes the Banach space of continuous functions in  $[0, t_0]$  and  $L^1([0, \infty))$  denotes the space of equivalent classes of Lebesgue integrable functions. We note that  $E$  is a Banach space.

In order to facilitate our future calculations, we need the following lemma:

**Lemma 3.1: 1** Suppose that  $x, y \geq 0$ , then  $|e^{-y} - e^{-x}| \leq |y - x|$ .

**Proof:** Let  $f(x) = e^{-x}$ , then use the mean value theorem to establish the required result.

Also, for the same purpose we note that by suitable changes of variables and reversing of the order of integration, we obtain that

$$\begin{aligned} & \int_t^\infty \int_0^t \int_0^\sigma e^{-\int_\sigma^t [\mu(a-t+\tau) + \gamma(a-t+\tau) + \varepsilon k_1(a-t+\tau)\lambda(\tau)] d\tau} k_1(a-t+\sigma)\lambda(\sigma)\delta(a-t+b) \times \\ & e^{-\int_b^\sigma [\mu(a-t+\tau) + \nu(a-t+\tau) + \delta(a-t+\tau) + k_1(a-t+\tau)\lambda(\tau)] d\tau} p(a-t+b, b) db d\sigma da \\ & + \int_0^t \int_0^a \int_0^\sigma e^{-\int_\sigma^a [\mu(\tau) + \gamma(\tau) + \varepsilon k_1(\tau)\lambda(t-a+\tau)] d\tau} k_1(\sigma)\lambda(t-a+\sigma)\delta(b) \times \\ & e^{-\int_b^\sigma [\mu(\tau) + \nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(t-a+\tau)] d\tau} p(b, t-a+b) db d\sigma da \\ & = \int_0^t \int_0^\infty \int_{t-b}^t e^{-\int_{a+\sigma-t}^a [\mu(\tau) + \gamma(\tau) + \varepsilon k_1(\tau)\lambda(\tau+t-a)] d\tau} k_1(a+b+\sigma-t)\lambda(\sigma)\delta(a) \times \\ & p(a, t-b) e^{-\int_a^{a+b+\sigma-t} [\mu(\tau) + \nu(\tau) + \delta(\tau) + k_1(\tau)\lambda(\tau+t-a-b)] d\tau} d\sigma dadb. \end{aligned} \quad (3.15)$$

In the next theorem, we prove the existence and uniqueness of solution to problem (2.1) via a fixed-point theorem.

**Theorem 3.1: 2** Problem (2.1) has a unique solution that exists for all time.

**Proof:** Define the set  $Q$  by  $Q = \{i(a, t) \in E, i(a, t) \geq 0, \|i(a, t)\| \leq M\}$ , where  $M$  is a constant which satisfies the following:

$$M > \|p_0(a)\|_{L^1} e^{(\|\beta(a)\|_\infty - \mu_*)t_0}. \quad (3.16)$$

We note that  $Q$  is a closed set in  $E$ . Now, for fixed initial age-distributions  $s_0(a), i_0(a), r_0(a)$  and  $p_0(a)$ , define the mapping  $T : Q \subset E \rightarrow E$  by







(2.1), and, in particular, we look at what is called disease-free equilibrium, and we show that it is possible that  $\lambda^* = 0$  and a positive endemic equilibrium exists, in fact, a continuum of positive endemic equilibria. This behavior is solely created by vertical transmission and is not present when  $q = 0$ .

**Theorem 4.1: 4** (1) Suppose that the following conditions hold:

- (i)  $k_2(a)$  is identically zero,
- (ii)  $q = 1$ , and (iii) the support of  $\gamma(a)$  lies to the right of the support of  $\beta(a)$ .

Then, problem (2.1) gives rise to a continuum of endemic equilibria of the form:

$$u^*(a) = (1 - v^*(0))e^{-\int_0^a [\nu(\tau) + \delta(\tau)] d\tau} + \int_0^a \delta(\sigma) e^{-\int_\sigma^a [\nu(\tau) + \delta(\tau)] d\tau} [1 - v^*(\sigma)] d\sigma, \quad (4.9)$$

$$v^*(a) = v^*(0) e^{-\int_0^a \gamma(\tau) d\tau}, \quad (4.10)$$

where the real number  $v^*(0) \in (0, 1]$ , is arbitrary.

- (2) If any one of the conditions in (1) does not hold then, if  $\lambda^* = 0$ , then the steady state of problem (2.1) is the disease-free equilibrium:

$$v^*(a) = 0, \\ u^*(a) = e^{-\int_0^a [\nu(\tau) + \delta(\tau)] d\tau} + \int_0^a \delta(\sigma) e^{-\int_\sigma^a [\nu(\tau) + \delta(\tau)] d\tau} d\sigma \quad (4.11)$$

- (3) Suppose that the following conditions hold:

- (i)  $k_2(a)$  is not identically zero,
- (ii)  $q = 1$ , and
- (iii) the supports of  $\gamma(a)$  and  $k_1(a)$  lie to the right of the support of  $\beta(a)$ . Then problem (2.1) gives rise to a continuum of endemic equilibria of the form:

$$u^*(a) = (1 - v^*(0))e^{-\int_0^a [\nu(\tau) + \delta(\tau) + \lambda^* k_1(\tau)] d\tau} + \int_0^a \delta(\sigma) e^{-\int_\sigma^a [\nu(\tau) + \delta(\tau) + \lambda^* k_1(\tau)] d\tau} [1 - v^*(\sigma)] d\sigma, \quad (4.12)$$

$$v^*(a) = v^*(0) e^{-\int_0^a [\gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} + \lambda^* \int_0^a k_1(\sigma) e^{-\int_\sigma^a [\gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} F(\sigma) d\sigma, \quad (4.13)$$

where the real number  $v^*(0) \in (0, 1]$ , is arbitrary, and  $F(\sigma)$  is defined as

$$F(\sigma) = [\varepsilon + (1 - \varepsilon)u^*(\sigma)]. \tag{4.14}$$

**Proof:** To prove (1), we note that if we solve the system of the ordinary differential equations (4.3)-(4.4), then we obtain (4.12)-(4.13). Then, if we set  $\lambda^* = 0$  in (4.13) and used the resulting equation and equation (4.4) to find  $v^*(0)$ , we find that  $v^*(0)$  is undetermined by (ii) and (iii).

Now, using equation (4.5), we obtain the following equation for  $\lambda^*$  and  $v^*(0)$  :

$$\begin{aligned} \lambda^* = & cv^*(0) \int_0^\infty k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau) + \varepsilon\lambda^*k_1(\tau)]d\tau} da \\ & + c\lambda^* \int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau) + \varepsilon\lambda^*k_1(\tau)]d\tau} F(\sigma)d\sigma da. \end{aligned} \tag{4.15}$$

Therefore, if we set  $\lambda^* = 0$ , then we see that  $v^*(0)$  is undetermined by (i). And accordingly, for arbitrary fixed  $v^*(0) \in (0,1]$ , we obtain an endemic equilibrium given by (4.9)-(4.10). This completes the proof of (1).

To prove (2), we note that it is easy to see that  $v^*(0) = 0$ , either from (4.15) and  $\lambda^* = 0$ , if  $k_2(a)$  is not identically zero or if any of the other conditions in (1) is not satisfied, then  $q \int_0^\infty \beta(a)\pi_2(a)da < 1$  and therefore, using (4.4), we see that  $v^*(0) = 0$ . Now, if we use equations (4.12)-(4.13), we obtain (4.11). This completes the proof of (2).

To prove (3), we note that using (4.4) and (4.13), it is easy to see that  $v^*(0)$  is undetermined, by assumptions (i)-(iii). Therefore, for arbitrary fixed  $v^*(0) \in (0,1]$ , we use equation (4.15) to determine at least one  $\lambda^* > 0$ . To this end, we rewrite equation (4.15) in the following form:

$$\begin{aligned} \lambda^* \left[ 1 - c \int_0^\infty \int_0^a k_2(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a [\gamma(\tau) + \varepsilon\lambda^*k_1(\tau)]d\tau} F(\sigma)d\sigma da \right] \\ = cv^*(0) \int_0^\infty k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau) + \varepsilon\lambda^*k_1(\tau)]d\tau} da. \end{aligned} \tag{4.16}$$

We can easily see that the right-hand side of (4.16) is a decreasing function of  $\lambda^*$ , with a value greater than zero when  $\lambda^* = 0$ , since  $k_2(a)$  is not identically zero, and tends to zero if  $\lambda^* \rightarrow +\infty$ . On the other hand, the left-hand side of (4.16) has a value equal to zero when  $\lambda^* = 0$  and approaches  $+\infty$ , when  $\lambda^* \rightarrow +\infty$ . Therefore, equation (4.16) has a solution  $\lambda^* > 0$ , and this value of  $\lambda^*$  gives rise to an endemic equilibrium via equations (4.12)-(4.13). Here, we note that the system of ODEs (4.3)-(4.4) has a unique solution for a fixed  $\lambda^*$  and a known  $v^*(0)$ . Also, we can easily see that  $u^*(a)$  satisfies the following integral equation, which has a unique solution for a fixed  $\lambda^*$  and a known  $v^*(0)$  :

$$u^*(a) = f(a; \lambda^*; v^*(0)) + \int_0^a K(s, a; \lambda^*)u^*(s)ds, \tag{4.17}$$

where  $f(a; \lambda^*; v^*(0))$  and  $K(s, a; \lambda^*)$  are functions that depend on the parameters of the



$$1 = c \int_0^\infty \int_0^a k_2(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_1(\sigma) F(\sigma) d\sigma da + \frac{cq \int_0^\infty \int_0^a \beta(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_1(\sigma) F(\sigma) d\sigma da \left[ \int_0^\infty k_2(a) \pi_2(a) e^{-\varepsilon \lambda^* \int_0^a k_1(\tau) d\tau} da \right]}{\left[ 1 - q \int_0^\infty \beta(a) \pi_2(a) e^{-\varepsilon \lambda^* \int_0^a k_1(\tau) d\tau} da \right]}, \quad (4.21)$$

where  $F(\sigma)$  is defined by equation (4.14).

Noticing that, if  $q \neq 1$ , i.e.,  $q < 1$  then from equation (4.4), we see that  $u^*(0) > 0$  and therefore,  $v^*(a) < 1 \forall a \in [0, \infty)$ . Accordingly, from equation (4.5), we see that  $\lambda^* < c \int_0^\infty k_2(a) \pi(a) da$ . Now, using this value for  $\lambda^*$  in equation (4.21) and the fact that  $v^*(a) < 1 \forall a \in [0, \infty)$ , and equation (4.13), we can deduce that the right-hand side of equation (4.21) is less than one at this value of  $\lambda^*$ . Also, it is easy to see that the right-hand side of (4.21) is equal to  $R_v > 1$ , when  $\lambda^* = 0$ . Therefore, equation (4.21) has at least one solution  $\lambda^* > 0$ . This completes the proof of the theorem.

Here, we note that from theorem (4.2), an endemic equilibrium would exist if  $R_v > 1$ , and from equation (4.6) the effect of vertical transmission via its parameter  $q$  which is the probability of vertically transmitting the disease, is seen  $R_v$  increases with  $q \in [0, 1)$ , and therefore increases the likelihood that an endemic will occur. So, in order to prevent an outbreak and control the spread of the disease, we need to reduce  $R_v$  to a value less than one. If  $v(a)$  is constant then there exists a unique value for  $v(a)$  which reduces  $R_v$  to one, but if  $v(a)$  is age-dependent and not constant then  $v(a)$  can be chosen according to some constraint that reduces the cost of vaccination or in general to obtain what is called an optimal vaccination strategy, for example, see Müller (1994), (1998), Haderler, et al. (1996), Castillo-Chavez, et al. (1998) and Li, et al. (2004).

If  $q = 1$ , and other conditions hold, for example, see Theorem (4.1), then problem (2.1) gives rise to a continuum of endemic steady states, and this situation does not occur when there is no vertical transmission. Also, if  $q = 1$ , and  $\gamma(a)$  is identically zero, then problem (2.1) has the steady state as the total population consisting of infective only. Also, it is easy to see that this steady state is actually, by uniqueness, the solution for problem (2.1), in this special case.

The effects of certain vaccination strategies for the eradication of important communicable diseases such as measles, rubella, pertussis and tuberculosis are dealt with in several papers, for example, see Hethcote (1983), (2000), (1997), (1989), Dietz (1981), Knox (1980), McLean (1986), Katzmann, et al. (1984), Schenzle (1984), Coutinho, et al. (1993), Greenhalgh (1990) and Anderson, et al (1999).

### 5. Stability of the Steady States

In this section, we study the stability of the steady states of problem (2.1), and, in particular, we study the stability of the disease-free equilibrium, and the endemic equilibria.







where  $F_2(a)$  is defined as

$$F_2(a) = \frac{(1-\varepsilon)\lambda^* f(a)}{\psi^*} + \left[ (1-\varepsilon)u^*(a) + \varepsilon(1-v^*(a)) \right]. \tag{5.12}$$

We note that if we set  $\lambda^* = 0$  in (5.11), we obtain (5.1).

In the following result, we establish the local asymptotic stability of an endemic equilibrium, in the special case  $\varepsilon = 1$ .

**Theorem 5.3:** 8 Suppose that  $R_v > 1$ ,  $q \neq 1$ , and  $\varepsilon = 1$ , then an endemic equilibrium is locally asymptotically stable.

**Proof:** We note that if  $\psi^* = 0$ , then  $g(a) = 0$  from equation (5.10), since  $g(0) = 0$  because  $\varepsilon = 1$ . Therefore, it follows from equation (5.9) that  $f(a) = 0$ , and hence, stability follows in this case. If  $\psi^* \neq 0$ , we take  $\xi$  in equation (5.11) to be real, and then it is clear from the characteristic equation (4.21) that  $\xi < 0$ , since  $v^*(a)$  satisfies equation (4.13), and  $R_v > 1$ , therefore the local asymptotic stability of an endemic equilibrium follows from the fact that the real root of the characteristic equation has the dominant real part. This completes the proof of the theorem.

In the next result, we will prove the local asymptotic stability of an endemic equilibrium, when  $q = 0$  and  $\delta(a)$  is identically zero. This result will allow us to deduce the stability of the endemic equilibrium of the SIR age-structured epidemic model, note that the endemic equilibrium is unique in this special case because of the monotonicity of the right-hand side of equation (4.21).

**Theorem 5.4:** 9 Suppose that  $R_v > 1$ ,  $q = 0$  and  $\delta(a)$  is identically zero. Then the endemic equilibrium is locally asymptotically stable.

**Proof:** Note that if  $\psi^* = q = 0$ , and  $\delta(a) \equiv 0$ , then it is easy to see that  $f(a) = 0$  from (5.9), since  $f(0) = -g(0) = 0$ . Accordingly, it follows that  $g(a) = 0$ . And hence, if  $\psi^* \neq 0$ , then the characteristic equation (5.11) takes the following form:

$$1 = c \int_0^\infty \int_0^a k_2(a)\pi(a) e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_1(\sigma) F_2(\sigma) d\sigma da, \tag{5.13}$$

where  $F_2(a)$  is given by equation (5.12).

Now, (5.13) can be rewritten in the following form:

$$1 = c \int_0^\infty \int_0^a k_2(a)\pi(a) k_1(\sigma) e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} \times \left\{ F(\sigma) - \left[ \lambda^* (1-\varepsilon) \int_0^\sigma e^{-\int_s^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(s) u^*(s) ds + \varepsilon v^*(\sigma) \right] \right\} d\sigma da, \tag{5.14}$$

where  $F(\sigma)$  is given by (4.14).

Therefore, if we assume that  $\xi$  is real in (5.14), then from equation (4.21), we conclude that  $\xi < 0$ , and accordingly, the local asymptotic stability follows from the fact that the real root of the characteristic equation has the dominant real part. This completes the proof of the theorem.

Note that if  $\delta(a) = q = \varepsilon = 0$ , then we obtain the SIR age-structured epidemic model, and theorem (5.4) shows that the endemic equilibrium of the SIR age-structured epidemic model is locally asymptotically stable.

In the next result, we prove the local asymptotic stability of an endemic equilibrium in the case  $q \neq 0$ , and  $\delta(a) \equiv 0$ .

**Theorem 5.5:** Suppose that the following hold:

(i)  $R_v > 1$ ,

(ii)  $\delta(a) \equiv 0$ , and

(iii)  $(1-\varepsilon)\lambda^* \int_0^\infty k_1(\sigma) e^{-\int_0^\sigma [\nu(\tau) + (1-\varepsilon)\lambda^* k_1(\tau) - \gamma(\tau)] d\tau} d\sigma < 1$ .

Then an endemic equilibrium is locally asymptotically stable.

**Proof:** If  $\psi^* = 0$ , then from equations (5.9)-(5.10), we obtain the following:

$$cg(0) \int_0^\infty k_2(a) \pi(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau) + \gamma(\tau)] d\tau} \times$$

$$[1 - (1-\varepsilon)\lambda^* \int_0^a k_1(\sigma) e^{-\int_0^\sigma [\nu(\tau) + (1-\varepsilon)\lambda^* k_1(\tau) - \gamma(\tau)] d\tau} d\sigma] da = 0. \quad (5.15)$$

From equation (5.15) and assumption (iii), we deduce that  $g(0) = 0$ , and then it follows that  $f(0) = 0$ , and from assumption (ii) and equation (5.9), we obtain that  $f(a) = 0$ . From  $f(a) = 0$  and equation (5.10), we obtain that  $g(a) = 0$ .

Now, if  $\psi^* \neq 0$ , then using equations (5.9)-(5.10) and solving for  $\frac{g(0)}{\psi^*}$ , we obtain the

following characteristic equation:

$$\left[ 1 - q \int_0^\infty \beta(a) \pi_2(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau)] d\tau} da \right] \times$$

$$\left\{ 1 - c \int_0^\infty \int_0^a k_2(a) \pi(a) e^{-\int_0^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_1(\sigma) F(\sigma) d\sigma da - \right.$$

$$c q \int_0^\infty \int_0^a \beta(a) \pi(a) e^{-\int_0^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_1(\sigma) F(\sigma) d\sigma da \left[ \int_0^\infty k_2(a) \pi_2(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau)] d\tau} da \right] \left. \right\}$$


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$$\left[ 1 - q \int_0^\infty \beta(a) \pi_2(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau)] d\tau} da \right]$$

$$\begin{aligned}
 & +c \left[ 1 - q \int_0^\infty \beta(a) \pi_2(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau)] d\tau} da \right] \times \\
 & \{ \varepsilon \int_0^\infty \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) k_1(\sigma) v^*(\sigma) d\sigma da + \lambda^* (1 - \varepsilon) \int_0^\infty \int_0^a \int_0^\sigma \times \\
 & e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) k_1(\sigma) k_1(s) u^*(s) e^{-\int_s^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} ds d\sigma da \} + \\
 & q \lambda^* (1 - \varepsilon) \int_0^\infty \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} \beta(a) \pi(a) k_1(\sigma) e^{-\int_0^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} d\sigma da \{ 1 - c \int_0^\infty \times \\
 & \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) k_1(\sigma) [F(\sigma) - \varepsilon v^*(\sigma)] d\sigma da + c \lambda^* (1 - \varepsilon) \int_0^\infty \int_0^a \int_0^\sigma \times \\
 & e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) k_1(\sigma) k_1(s) u^*(s) e^{-\int_s^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} ds d\sigma da \} = -\varepsilon q c \int_0^\infty \times \\
 & \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} \beta(a) \pi(a) k_1(\sigma) v^*(\sigma) d\sigma da \left( \int_0^\infty k_2(a) \pi_2(a) e^{-\int_0^a [\xi + \varepsilon \lambda^* k_1(\tau)] d\tau} da \right) \\
 & - q c \lambda^* (1 - \varepsilon) \int_0^\infty \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) k_1(\sigma) e^{-\int_0^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} d\sigma da \times \\
 & \left( \int_0^\infty \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} \beta(a) \pi(a) k_1(\sigma) [F(\sigma) - \varepsilon v^*(\sigma)] d\sigma da \right) - q c \lambda^* (1 - \varepsilon) \times \\
 & \int_0^\infty \int_0^a e^{-\int_\sigma^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} \beta(a) \pi(a) k_1(\sigma) \left( \int_0^\sigma e^{-\int_s^\sigma [\xi + \nu(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(s) u^*(s) ds \right) d\sigma da \left( \int_0^\infty \times \right. \\
 & \left. e^{-\int_0^a [\xi + \gamma(\tau) + \varepsilon \lambda^* k_1(\tau)] d\tau} k_2(a) \pi(a) [1 - \lambda^* (1 - \varepsilon) \int_0^a k_1(\sigma) e^{-\int_0^\sigma [\nu(\tau) + (1 - \varepsilon) \lambda^* k_1(\tau) - \gamma(\tau)] d\tau} d\sigma] da \right).
 \end{aligned}
 \tag{5.16}$$

Now, suppose that  $\xi$  is real, then from the characteristic equation (5.16) and equation (4.21), and the fact that  $R_\nu > 1$ , we obtain that  $\xi < 0$ , since for  $\xi \geq 0$ , the right-hand side of (5.16) is non-positive while the left-hand side is positive. Therefore, an endemic equilibrium is locally asymptotically stable, since the real root has the dominant real part. This completes the proof of the theorem.

We note that if, in addition to the assumptions of Theorem (5.5), we assume that  $\varepsilon = 0$  in the model, then we obtain the SIR age-structured epidemic model studied in Cha, et al. (1998) and therefore, Theorem (5.5) establishes the local asymptotic stability for the endemic equilibrium of that model. Also, we note that if  $\delta(a) \equiv 0$ , and  $\varepsilon = 0$ , then condition (iii) of Theorem (5.5) guarantees the uniqueness of the endemic equilibrium, and this follows from the monotonicity of the right-hand side of equation (4.21).

Now, we look at the case in which  $\delta(a)$  is not identically zero but  $q = 0$ , for example, see Li, et al. (2004). In this case, we first use the system of ODEs (5.9)-(5.10) to obtain that if  $\psi^* = 0$ , then  $f(a) = g(a) = 0$ , and that is because  $f(0) = g(0) = 0$  and the uniqueness of solution of the system of ODEs. Also, if we assume that  $\psi^* \neq 0$ , then using the same system of equations, we can deduce that  $G(a) = \frac{g(a)}{\psi^*}$  satisfies the following Volterra integral equation:

$$G(a) = \int_0^a K(a, s)G(s)ds + d(a), \quad (5.17)$$

where  $K(a, s)$  and  $d(a)$  are given by

$$K(a, s) = -\lambda^* (1-\varepsilon)\delta(s) \int_s^a k_1(\sigma) e^{-\int_s^a [\xi + \gamma(\tau) + \varepsilon\lambda^* k_1(\tau)] d\tau} e^{-\int_s^\sigma [\xi + \nu(\tau) + \delta(\tau) + (1-\varepsilon)\lambda^* k_1(\tau) - \gamma(\tau)] d\tau} d\sigma, \quad (5.18)$$

$$\begin{aligned} d(a) = & -\lambda^* (1-\varepsilon) \int_0^a e^{-\int_0^a [\xi + \gamma(\tau) + \varepsilon\lambda^* k_1(\tau)] d\tau} k_1(\sigma) \left( \int_0^\sigma e^{-\int_0^\sigma [\xi + \nu(\tau) + \delta(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(s) u^*(s) ds \right) d\sigma \\ & + \int_0^a e^{-\int_0^a [\xi + \gamma(\tau) + \varepsilon\lambda^* k_1(\tau)] d\tau} k_1(\sigma) [F(\sigma) - \varepsilon v^*(\sigma)] d\sigma. \end{aligned} \quad (5.19)$$

Also, in this case the characteristic equation is given by the following equation:

$$1 = c \int_0^\infty G(a) k_2(a) \pi(a) da. \quad (5.20)$$

We note that the Volterra integral equation (5.17) has a unique solution, and this fact can also be seen from the system of ODEs (5.9)-(5.10).

In the next result, we show that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

**Theorem 5.6:** Suppose that  $R_v > 1$ , and  $q = 0$ . If  $\lambda^*$  is sufficiently small, then an endemic equilibrium is locally asymptotically stable.

**Proof:** We note that  $G(a) = \frac{g(a)}{\psi^*} = e^{-\xi t} \frac{[\nu(a, t) - v^*(a)]}{\psi^*}$ , and so for  $\text{Re } \xi \geq 0$  and  $\psi^* \neq 0$ ,  $|G(a)|$  is bounded and its integral from 0 to  $\infty$  is also bounded, because we are assuming that the total population has already reached its steady state. Now, since  $\varepsilon > 0$ , we can use equation (4.21) to show that the following integral is bounded:

$$c \int_0^\infty \int_0^a k_2(a) \pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \varepsilon\lambda^* k_1(\tau)] d\tau} k_1(\sigma) d\sigma da. \quad (5.21)$$

Using equations (5.17) and (5.20), we obtain that

$$1 = c \int_0^\infty \int_0^a K(a, s) k_2(a) \pi(a) G(s) ds da + c \int_0^\infty k_2(a) \pi(a) d(a) da. \quad (5.22)$$

Therefore, from equations (5.18)-(5.19), we obtain that the right-hand side of equation (5.22) approaches

$$c \int_0^\infty \int_0^a k_2(a) \pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \varepsilon\lambda^* k_1(\tau)] d\tau} k_1(\sigma) [F(\sigma) - \varepsilon v^*(\sigma)] d\sigma da \text{ as } \lambda^* \rightarrow 0. \quad (5.23)$$

And therefore, in view of equations (4.21) and (4.13), the right-hand side of equation (5.22) is strictly less than one for any  $\xi$  with  $\text{Re } \xi \geq 0$ , and  $\lambda^*$  sufficiently small. Accordingly, an endemic equilibrium is locally asymptotically stable if  $\lambda^*$  is sufficiently small. This completes the proof of the theorem.



asymptotically stable, if the force of infection is sufficiently small.

**Theorem 5.7:** 12 Suppose that  $R_v > 1$ , and  $k_2(a)$  and  $\beta(a)$  are constants independent of age. If  $\lambda^*$  is sufficiently small, then an endemic equilibrium is locally asymptotically stable.

**Proof:** The proof of this theorem is similar to the proof of Theorem (5.6), and therefore, we omit the proof.

## 6. Conclusion

We studied an age-structured SIRS epidemic model, when the disease is vertically as well as horizontally transmitted and the force of infection of proportionate mixing assumption type, susceptible individuals are vaccinated with vaccine that wanes over time, and therefore immune individuals are susceptible with some resistance to the disease. The mortality and fertility rates are age-dependent.

We proved the existence and uniqueness of solution to the model equations and showed that solutions of the model equations depend continuously on the initial age-distributions, and therefore, the well posedness of the problem is proved.

Furthermore, we determined the steady states of the model and examined their stability, when  $q \neq 1$ , by determining an explicitly computable threshold parameter  $R_v$ , in terms of the demographic and epidemiological parameters of the model, known as the reproduction number in the presence of vaccination strategy  $v(a)$ , as in Haderler, et al. (1996) or the net replacement ratio, as in Thieme (2001). And hence, we showed that  $R_v$  increases with  $q$ , which is the probability of vertically transmitting the disease, and therefore, increases the likelihood that an endemic will occur; also  $R_v$  decreases with  $v(a)$ , and is used to determine a critical vaccination coverage which will eradicate the disease with minimum vaccination coverage.

If  $R_v \leq 1$ , then the only steady state of problem (2.1) is the disease-free equilibrium, and is locally asymptotically stable if  $R_v < 1$ , and globally stable if  $R_0 < 1$ . If  $R_v > 1$ , then a disease-free equilibrium and an endemic equilibrium are possible steady states, the disease-free equilibrium is unstable. The question of uniqueness of an endemic equilibrium is an open problem, and this is also the case for several age-structured epidemic models, for example, see Castillo-Chavez, et al. (1998), Cha, et al. (1998), and El-Doma (2006).

Concerning the stability of an endemic equilibrium, we obtained a complicated characteristic equation that allowed to prove local asymptotic stability, in some special cases, which are: (i) the case  $\varepsilon = 1$ , (ii) the case  $q = 0$ , and  $\delta \equiv 0$ , (iii) the case  $\delta \equiv 0$ , and

$(1-\varepsilon)\lambda^* \int_0^\infty k_1(\sigma) e^{-\int_0^\sigma [v(\tau) + (1-\varepsilon)\lambda^* k_1(\tau) - \gamma(\tau)] d\tau} d\sigma < 1$ . And we derived general formulas for the characteristic equations, in terms of integral equations. Although explicit formulas for the solutions of these integral equations are not obtained, we are able to use them to deduce that if the force of infection is sufficiently small, then an endemic equilibrium is always locally asymptotically stable.

If  $q = 1$ , and other conditions hold, for example, see Theorem (4.1), then problem (2.1) gives rise to a continuum of endemic steady states, and this situation does not occur when there is no vertical transmission. Also, if  $q = 1$ , and  $\gamma(a)$  is identically zero, then problem (2.1) has the total population consisting of infective individuals only as the steady state.

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