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Fuzzy Efficiency Measure with Fuzzy Production Possibility Set

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Abstract

The existing data envelopment analysis (*DEA*) models for measuring the relative efficiencies of a set of decision making units (*DMU*s) using various inputs to produce various outputs are limited to crisp data. The notion of fuzziness has been introduced to deal with imprecise data. Fuzzy *DEA* models are made more powerful for applications. This paper develops the measure of efficiencies in input oriented of *DMU*s by envelopment form in fuzzy production possibility set (*FPPS*) with constant return to scale.

Keywords: Data Envelopment Analysis; Fuzzy Number; $\alpha-$cut.

MSC 2000; 41A30, 65L06

1. Introduction

Data Envelopment Analysis (*DEA*) was suggested with *CCR* model by Charnes et al. (1978) and built on the idea of Farrell (1957) which is concerned with the estimation of technical efficiency and efficient frontiers. Several models introduced for evaluating of efficiency such as Charnes et al. (1978) Banker et al. (1984) and Cooper et al. (1999). Most of practical data in such situations economic evaluating, are imprecise. To deal quantitatively with imprecision in decision progress, Bellman et al. (1970) introduce the notion of fuzziness. Some researchers have proposed several fuzzy models to evaluate DMUs with fuzzy data, without access to *FPPS* (see Chiang et al. (2000) Lertworasirikul et al. (2003) Wang et al. (2005) Tanaka et al. (2001), Jahanshahloo et al. (2004) Leon et al. (2003) Kao et al. (2003)). Unfortunately, some of the existing techniques only provide crisp solution. Lertworasirikul et al. (2003) provided fuzzy efficiency measures in multiple models.

In this paper, fuzzy efficiency measures with fuzzy production, possibly set (*FPPS*) in constant return to scale with input oriented, such that it satisfies the initial concept by crisp data is proposed. Data envelopment analysis and fuzzy system are considered in section (2) and (3), respectively. Subsequently, production possibility set (*PPS*) to be extended by fuzzy data and then the proposed fuzzy efficiency measures in *CCR* model are discussed in section (4). A numerical example is presented in section (5). Finally, we conclude the paper in section (6).

2. Data Envelopment Analysis (DEA)

DEA utilizes a technique of mathematical programming for evaluation of n DMUs . Suppose there are n DMUs: $\text{DMU}_1, \text{DMU}_2, ..., \text{DMU}_n$. Let the input and output data for DMU_j be $X_j = (x_{1j}, ..., x_{mj})$ and $Y_j = (y_{1j}, ..., y_{sj})$, respectively.

2.1 Production Possibility Set (PPS)

We will call a pair of input $X \in R^m$ and output $Y \in R^s$ an activity and express them by the notation (*X*,*Y*). The set of feasible activities is called the production possibility set (*PPS*) and is denoted by *P*. We postulate the following properties of *P*:

1: The observed activities $(X_j\ Y_j) \in P;\ j = 1,..., n$

2: If an activity $(X, Y) \in P$, then the activity $(tX, tY) \in P$ for all $t > 0$.

3: If an activity $(X, Y) \in P$, then $(\bar{X}, \bar{Y}) \in P$ if $\bar{X} \geq X$ and $\bar{Y} \leq Y$.

4: If activity $(X, Y) \in P$ and $(\bar{X}, \bar{Y}) \in P$, then $(\lambda X + (1-\lambda)\bar{X}, \lambda Y + (1-\lambda)\bar{Y}) \in P$ for all¸ $\lambda \in [0,1]$.

We show the set *P* as follow:

$$P = \left\{ (x, y) \middle| x \geq \sum_{j=1}^{n} \lambda_j x_j,\ y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0;\ j=1,...,n \right\}.$$

2.2 The CCR Model

The CCR model proposed by Charnes et al. (1978) is as follow:

$$
\begin{array}{lllll}
\min & \theta & & & \\
\text{s.t} & \theta x_0 & \geq & \sum_{j=1}^{n}\lambda_j x_j, & j=1,\dots,n \\
 & y_0 & \leq & \sum_{j=1}^{n}\lambda_j y_j, & j=1,\dots,n \\
 & \lambda_j & \geq & 0, & j=1,\dots,n
\end{array} \qquad (1)
$$

The constructions of *CCR* model require the activity $(\theta x_0, y_0) \in PPS$, while the objective seeks the min θ that reduces the input vector x_0 radially to θx_0 while remaining in *PPS*. In *CCR* model, we are looking for an activity in PPS that guarantees at least the output level y_0 of DMU_0 in all components while reducing the input vector x_0 proportionally (radially) to a value as small as possible.

3. Fuzzy Systems

Let X be a nonempty set. A fuzzy set $\tilde{A}$ in X is characterized by its membership function $\mu_{\tilde{A}}: X \to [0,1]$ and $\mu_{\tilde{A}}(x)$ is interpreted as the degree of membership of element X in fuzzy set $\tilde{A}$ for each $x \in X$. A fuzzy set $\tilde{A}$ is completely determined by the set of tuples $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$.

Definition 1. An $\alpha-$level set of a fuzzy $\tilde{A}$ of X is a non-fuzzy set denoted by $[\tilde{A}]^{\alpha}$ and is defined by $[\tilde{A}]^{\alpha} = [A^{l\alpha}, A^{u\alpha}]$ where $A^{l\alpha} = \min\{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$ and $A^{u\alpha} = \max\{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 2. A fuzzy number $\tilde{A}$ is a fuzzy set of the real line with a normal, (fuzzy) convex and continuous membership function of bounded support.

Definition 3. If $[A^{l\alpha}, A^{u\alpha}]$ and $[B^{l\alpha}, B^{u\alpha}]$ are $\alpha-$levels of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ respectively and $\lambda \in R$ then:

1- $[A^{l\alpha}, A^{u\alpha}] + [B^{l\alpha}, B^{u\alpha}] = [A^{l\alpha} + B^{l\alpha}, A^{u\alpha} + B^{u\alpha}]$

2- $\lambda[A^{l\alpha}, A^{u\alpha}] = \begin{cases} [\lambda A^{l\alpha}, \lambda A^{u\alpha}], & if\ \lambda \geq 0 \\ [\lambda A^{u\alpha}, \lambda A^{l\alpha}], & if\ \lambda < 0 \end{cases}$

3.1 Extension Principle

Let X be a Cartesian product of universes $X = X_1 \times X_2 \times ... \times X_r$ and $\tilde{A}_1, ..., \tilde{A}_r$ be r fuzzy sets in $X_1, X_2, ..., X_r$ respectively. f is a mapping from X to a universe Y, $y = f(x_1, ..., x_r)$. Then the extension principle allows us to define fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = \left\{(y, \mu_B(y)) \middle| y = f(x_1, ..., x_r), (x_1, ..., x_r) \in X_1 \times ... \times X_r \right\},$$

where

$$\mu_B(y) = \begin{cases} \sup\limits_{(x_1, ..., x_r) \in f^{-1}(y)} \min\left\{\mu_{A_1}(x_1), ..., \mu_{A_r}(x_r)\right\} & f^{-1}(y) \neq \phi \\ 0 & \textit{otherwise}, \end{cases}$$

where f^{-1} is the inverse of f .

4. Fuzzy Efficiency Measure

In this section we are going to obtain fuzzy efficiency measure in *FPPS*.

4.1 Fuzzy Production Possibility Set (FPPS)

Let the input and output data for *DMUj* (j = 1, ..., n) be $\tilde{X}_j = (\tilde{x}_{1j}, ..., \tilde{x}_{mj})^t$ and $\tilde{Y}_j = (\tilde{y}_{1j}, ..., \tilde{y}_{sj})^t$, respectively, such as $\tilde{x}_{ij} (i = 1, ..., m)$ and $\tilde{y}_{rj} (r = 1, ..., s)$ for $j = 1, ..., n$ be $(m + s)n$ fuzzy numbers. We call the set of feasible activities with fuzzy data fuzzy production possibility set (*FPPS*) and denote it by $\tilde{P}$.

We define the fuzzy production possibility set (*FPPS*) as follow:

$$\tilde{P} = \{((X, Y), \mu_{\tilde{P}}(X, Y)) \mid X \in R^m, \quad Y \in R^s\}, \tag{2}$$

where

$$X = (x_1, ..., x_m) \ , \quad (x_i, \mu_{\tilde{x}_i}(x_i)) \in \tilde{x}_i \quad i = 1, ...m \tag{3}$$

$$Y = (y_1, ..., y_s) \ , \quad (y_i, \mu_{\tilde{y}_i}(y_i)) \in \tilde{y}_i \quad i = 1, ...s. \tag{4}$$

Then, with extension principle

$$\begin{aligned} \mu_{\tilde{P}}(X,Y) = \max \quad & \min_{\lambda_j>0} \{\mu_{\tilde{x}_{1j}}(x_{1j})\ldots,\mu_{\tilde{x}_{mj}}(x_{mj}),\mu_{\tilde{y}_{1j}}(y_{1j}),\ldots,\mu_{\tilde{y}_{sj}}(y_{sj})\} \\ s.t \quad & X \geq \sum_{j=1}^{n} \lambda_j X_j \\ & Y \leq \sum_{j=1}^{n} \lambda_j Y_j \\ & \lambda_j \geq 0,\ j=1,\ldots,n \\ & (X_j,Y_j) \in \operatorname{supp}(\tilde{X}_j,\tilde{Y}_j) \end{aligned} \tag{5}$$

such as $(\tilde{X}_j,\tilde{Y}_j)$ $(j=1,2,\ldots,n)$ is observed activities.

Notation 1. Let S be set of vectors then $\bar{X}=\min S$ if and only if $\forall X \in S; X \geq \bar{X}$**.**

Notation 2. We define $FPPS^{\beta}$ as follow:

$$FPPS^{\beta} = \left\{ (X,Y) \;\middle|\; \begin{array}{l} ((X,Y)\ ,\ \mu_{\tilde{P}}(X,Y)) \in \tilde{P} \\ \mu_{\tilde{P}}(X,Y) \geq \beta \end{array} \right\} = [\tilde{P}]^{\beta}. \tag{6}$$

The CCR model by fuzzy data require the activity $(\tilde{\theta}\tilde{X}_0,\tilde{Y}_0)$ to belong to $\tilde{P}$, while for any $((X_0,Y_0),\alpha)\in(\tilde{X}_0,\tilde{Y}_0)$ and $0<\alpha\leq 1$ that $\mu_{\tilde{P}}(X_0,Y_0)\geq\beta$ the objective seeks the minimum $\theta(\theta\in\tilde{\theta})$ that reduces input vector X_0 radially to θX_0 while $(\theta X_0,Y_0) \in \operatorname{supp}\tilde{P}^{\beta}$. Hence, CCR model is proposed as follows:

$$\begin{aligned} \min \quad & \tilde{\theta} \\ s.t \quad & (\tilde{\theta}\tilde{X}_0,\tilde{Y}_0) \in \tilde{P} \end{aligned} \tag{7}$$

But

$$(\tilde{\theta}\tilde{X}_0,\tilde{Y}_0) \in \tilde{P} \ \ \textit{if and only if} \ \begin{cases} [(\tilde{\theta}\tilde{X}_0,\tilde{Y}_0)]^{\alpha} \subseteq [\tilde{P}]^{\beta} \\ 0<\alpha\leq 1 \\ 0<\beta\leq 1 \\ (X_0,Y_0)\in[\tilde{P}]^{\beta} \ \textit{for any}\ (X_0,Y_0)\in[(\tilde{X}_0,\tilde{Y}_0)]^{\alpha} \end{cases}$$

If $[(\tilde{\theta}\tilde{X}_0,\tilde{Y}_0)]^{\alpha}=[E_1,E_2]$, then with (7) and notation 1 $\theta^{l\alpha}$ is the efficiency measure with maximum input and minimum output. Hence,

$$E_1=\min\{(\theta X_0,Y_0)|\theta\in[\tilde{\theta}]^{\alpha},X_0\in[\tilde{X}_0]^{\alpha},Y_0\in[\tilde{Y}_0]^{\alpha}\}=(\theta^{l\alpha}X_0^{u\alpha},Y_0^{l\alpha}).$$

Similarity, $\theta^{u\alpha}$ is the efficiency measure with minimum input and maximum output. Hence,

$$E_2 = \max\{\theta X_0, Y_0) | \theta \in [\tilde{\theta}]^{\alpha}, X_0 \in [\tilde{X}_0]^{\alpha}, Y_0 \in [\tilde{Y}_0]^{\alpha}\} = (\theta^{u\alpha} X_0^{la}, Y_0^{u\alpha})$$

Therefore,

$$(\tilde{\theta}\tilde{X}_0, \tilde{Y}_0) \in \tilde{P} \ \ \textit{if and only if} \begin{cases} (\theta^{l\alpha} X_0^{u\alpha}, Y_0^{l\alpha}) \in [\tilde{P}]^{\beta} \\ (\theta^{u\alpha} X_0^{l\alpha}, Y_0^{u\alpha}) \in [\tilde{P}]^{\beta} \\ 0 < \alpha \le 1 \\ 0 < \beta \le 1 \\ (X_0^{u\alpha}, Y_0^{l\alpha}) \in [\tilde{P}]^{\beta} \\ (X_0^{l\alpha}, Y_0^{u\alpha}) \in [\tilde{P}]^{\beta} \end{cases}$$

Let, for any $(X_0, Y_0) \in \left[(\tilde{X}_0, \tilde{Y}_0)\right]^{\alpha}$ with $\mu_{\tilde{P}}(X_0, Y_0) \ge \beta$, $[\tilde{\theta}]^{(\alpha,\beta)} = [\theta^{l(\alpha,\beta)}, \theta^{u(\alpha,\beta)}]$ such that $\mu_{\tilde{P}}(\theta X_0, Y_0) \ge \beta$ for all $\theta \in [\theta^{l(\alpha,\beta)}, \theta^{u(\alpha,\beta)}]$.($\alpha \in \left[0,1\right]$ is a membership function of fuzzy DMU and $\beta \in \left[0,1\right]$ is a membership function of production frontier). Hence:

$$\begin{aligned} \theta^{l(\alpha,\beta)} = \ \min \quad & \theta \\ s.t \quad & \theta X_0^{u\alpha} \ge \sum_{j=1}^{n} \lambda_j X_j \\ & Y_0^{l\alpha} \le \sum_{j=1}^{n} \lambda_j Y_j \\ & X_0^{u\alpha} \ge \sum_{j=1}^{n} \lambda_j X_j \\ & \lambda_j \ \ge 0, \ j = 1, \ldots, n \\ (X_j, Y_j) \in [(\tilde{X}_j, \tilde{Y}_j)]^{\beta} & \end{aligned} \tag{8}$$

$$
\begin{aligned}
\theta^{u(\alpha,\beta)} = \min \quad & \theta \\
s.t \quad & \theta X_0^{l\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j \\
& Y_0^{u\alpha} \leq \sum_{j=1}^{n} \lambda_j Y_j \\
& X_0^{l\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j \\
& \lambda_j \geq 0, \qquad j=1,\ldots,n \\
& (X_j, Y_j) \in [(\tilde{X}_j, \tilde{Y}_j)]^{\beta}
\end{aligned}
\tag{9}
$$

The lowest and highest relative efficiency of DMU_0 is found by proposed models as follows:

$$
\begin{aligned}
\theta'^{l}(\alpha,\beta) = \min \quad & \theta \\
s.t \quad & \theta X_0^{u\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\
& Y_0^{l\alpha} \leq \sum_{j=1}^{n} \lambda_j Y_j^{u\beta} + \sum_{j=1}^{n} \mu_j Y_j^{l\beta} \\
& X_0^{u\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\
& \lambda_j \geq 0, \quad j=1,\ldots,n \\
& \mu_j \geq 0, \quad j=1,\ldots,n
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\theta'^{u}(\alpha,\beta) = \min \quad & \theta \\
s.t \quad & \theta X_0^{l\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j^{u\beta} + \sum_{j=1}^{n} \mu_j X_j^{l\beta} \\
& Y_0^{u\alpha} \leq \sum_{j=1}^{n} \lambda_j Y_j^{l\beta} + \sum_{j=1}^{n} \mu_j Y_j^{u\beta} \\
& X_0^{l\alpha} \geq \sum_{j=1}^{n} \lambda_j X_j^{u\beta} + \sum_{j=1}^{n} \mu_j X_j^{l\beta} \\
& \lambda_j \geq 0 \qquad j=1,\ldots,n \\
& \mu_j \geq 0 \qquad j=1,\ldots,n
\end{aligned}
\tag{11}
$$

Theorem 1. Model (10) is feasible always if $\alpha \geq \beta$**.**

Theorem 2. Model (11) is feasible always if $\alpha \geq \beta$**.**

Theorem 3. $\theta^{l(\alpha,\beta)} = \theta'^{l(\alpha,\beta)}$.

Proof: Let, $P^{\beta}(8)$ and $P^{\beta}(10)$ be production possibility sets of models (8) and (10), respectively. We show that production possibility set of model (8) is equal to production possibility set of model (10).

$$P^{\beta}(10) = \left\{ (X,Y): \begin{array}{ll} X \geq \sum_{j=1}^{n} \lambda_j \overset{l\beta}{X}_j + \sum_{j=1}^{n} \mu_j \overset{u\beta}{X}_j & : \quad \lambda_j \geq 0, j=1,\ldots,n \\ \\ Y \leq \sum_{j=1}^{n} \lambda_j \overset{u\beta}{Y}_j + \sum_{j=1}^{n} \mu_j \overset{l\beta}{Y}_j & : \quad \mu_j \geq 0, j=1,\ldots,n \end{array} \right\}$$

$$P^{\beta}(8) = \left\{ (X,Y): \begin{array}{ll} X \geq \sum_{j=1}^{n} \lambda_j X_j; & \lambda_j \geq 0, j=1,\ldots,n, \\ \\ Y \leq \sum_{j=1}^{n} \lambda_j Y_j; & (X_j, Y_j) \in [(\tilde{X}_j, \tilde{Y}_j)]^{\beta} \end{array} \right\}$$

$P^{\beta}(10) \subseteq P^{\beta}(8)$ because if $(X,Y) \in P^{\beta}(10)$ then

$$\begin{cases} X \geq \sum_{j=1}^{n} \lambda_j \overset{l\beta}{X}_j + \sum_{j=1}^{n} \mu_j \overset{u\beta}{X}_j \\ Y \leq \sum_{j=1}^{n} \lambda_j \overset{u\beta}{Y}_j + \sum_{j=1}^{n} \mu_j \overset{l\beta}{Y}_j \end{cases}$$

If $\lambda_j + \mu_j = k_j$, then $\begin{cases} \lambda_j = a_j . k_j \\ \mu_j = b_j . k_j \\ a_j + b_j = 1 \end{cases} \Rightarrow \begin{cases} X \geq \sum_{j=1}^{n} k_j X_j \\ Y \leq \sum_{j=1}^{n} k_j Y \\ (X_j, Y_j) \in [(\tilde{X}_j, \tilde{Y}_j)]^{\beta}, j=1,\ldots,n. \end{cases}$

Hence, $(X, Y) \in P^{\beta}(8)$.

Let $P^{\beta}(8) \not\subseteq P^{\beta}(10)$ then $\exists (X,Y);(X,Y) \in P^{\beta}(8)$ and $(X,Y) \notin P^{\beta}(10)$. Hence, one of the three cases has occurred:

$$\forall \lambda_j, \mu_j \begin{cases} X < \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\ Y \le \sum_{j=1}^{n} \lambda_j Y_j^{u\beta} + \sum_{j=1}^{n} \mu_j Y_j^{l\beta} \end{cases} \quad or \quad \begin{cases} X \ge \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\ Y > \sum_{j=1}^{n} \lambda_j Y_j^{u\beta} + \sum_{j=1}^{n} \mu_j Y_j^{l\beta} \end{cases} \quad or$$

$$\begin{cases} X < \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\ Y > \sum_{j=1}^{n} \lambda_j Y_j^{u\beta} + \sum_{j=1}^{n} \mu_j Y_j^{l\beta} \end{cases}$$

Let,

$$\begin{cases} X < \sum_{j=1}^{n} \lambda_j X_j^{l\beta} + \sum_{j=1}^{n} \mu_j X_j^{u\beta} \\ Y > \sum_{j=1}^{n} \lambda_j Y_j^{u\beta} + \sum_{j=1}^{n} \mu_j Y_j^{l\beta} \end{cases}$$

If $\lambda_j + \mu_j = k_j$, then

$$\begin{cases} \lambda_j = a_j . k_j \\ \mu_j = b_j . k_j \\ a_j + b_j = 1 \end{cases} \Rightarrow \begin{cases} X < \sum_{j=1}^{n} k_j X_j \\ Y > \sum_{j=1}^{n} k_j Y_j \\ (X_j, Y_j) \in [(\tilde{X}_j, \tilde{Y}_j)]^{\beta}, j = 1, \dots, n \end{cases}$$

Hence, $(X,Y) \notin P^{\beta}(10)$ which regarding to assumption this isn't correct. Hence, proof is completed.

Theorem 4. $\theta^{u(\alpha,\beta)} = \theta'^{u(\alpha,\beta)}$.

Proof: Similar to proof of Theorem 1.

Lemma 1. If $\theta'^{l(\alpha,\beta)}$ and $\theta'^{u(\alpha,\beta)}$ are optimal solutions of models (10) and (11), respectively, then, $\theta'^{l(\alpha,\beta)} \leq \theta'^{u(\alpha,\beta)}$.

The equality is held when all component of DMU_0 are generated from fuzzy data to exact data.

Proof: Regarding to theorem 1 and theorem 2 and the models (8) and (9) proof is evident ($\theta'^{u(\alpha,\beta)}$ is a feasible solution of model (8)).

Lemma 2. If $\alpha_1 \leq \alpha_2$ then $\theta'^{l(\alpha_1,\beta)} \leq \theta'^{l(\alpha_2,\beta)}$ and $\theta'^{u(\alpha_2,\beta)} \leq \theta'^{u(\alpha_1,\beta)}$ for all $\beta \in (0,1]$ in both models (10) and (11).

Proof: If β is constant and $\alpha_1 \leq \alpha_2$ then $Y_0^{l\alpha_1} \leq Y_0^{l\alpha_2}$ and $X_0^{u\alpha_1} \geq X_0^{u\alpha_2}$ hence regarding to model (10) $\theta'^{l(\alpha_1,\beta)} \leq \theta'^{l(\alpha_2,\beta)}$. Similarly, if β is constant and $\alpha_1 \leq \alpha_2$ then $X_0^{l\alpha_1} \leq X_0^{l\alpha_2}$ and $Y_0^{u\alpha_1} \geq Y_0^{u\alpha_2}$ hence regarding with model (11) $\theta'^{u(\alpha_2,\beta)} \leq \theta'^{u(\alpha_1,\beta)}$.

Lemma 3. If $\beta_1 \leq \beta_2$ then $\theta'^{l(\alpha,\beta_1)} \leq \theta'^{l(\alpha,\beta_2)}$ and $\theta'^{u(\alpha,\beta_1)} \leq \theta'^{u(\alpha,\beta_2)}$ for all $\alpha \in (0,1]$ in both models (10) and (11).

Proof: If α is constant and $\beta_1 \leq \beta_2$ then $\left[(\tilde{X}_j,\tilde{Y}_j)\right]^{\beta_2} \subseteq \left[(\tilde{X}_j,\tilde{Y}_j)\right]^{\beta_1}$. Hence, regarding to theorem 1 and model (8) $\theta'^{l(\alpha,\beta_1)} \leq \theta'^{l(\alpha,\beta_2)}$. Similarly, if α is constant and $\beta_1 \leq \beta_2$, then $[(\tilde{X}_j,\tilde{Y}_j)]^{\beta_2} \subseteq [(\tilde{X}_j,\tilde{Y}_j)]^{\beta_1}$. Hence, regarding to theorem 2 and model (9) $\theta'^{u(\alpha,\beta_1)} \leq \theta'^{u(\alpha,\beta_2)}$.

Proposition 1. Model (10) is non-feasible iff $(X_0^{u\alpha}, Y_0^{l\alpha}) \notin \tilde{P}^{\beta}$.

Proposition 2. Model (11) is non-feasible iff $(X_0^{l\alpha}, Y_0^{u\alpha}) \notin \tilde{P}^{\beta}$.

Proposition 3. If both models (10) and (11) are feasible $([(\tilde{X},\tilde{Y})]^{\alpha} \subseteq \tilde{P}^{\beta})$, then $\theta'^{l(\alpha,\beta)} \leq 1$ and $\theta'^{u(\alpha,\beta)} \leq 1$.

5. Numerical Example

A simple numerical example with fuzzy single-input and single-output was introduced by C. Kao and S.T. Liu (2000). We will consider this example with its data listed in table 1. These DMUs (A,B,C and D) are evaluated by proposed models in (10) and (11).

Table 1:

DMU_s	Input	$\alpha-cut$	Output	$\alpha-cut$
A	(11,12,14)	$[11+\alpha, 14-2\alpha]$	(10,10,10)	[10,10]
B	(30,30,30)	[30,30]	(12,13,14,16)	$[12+\alpha, 16-2\alpha]$
C	(40,40,40)	[40,40]	(11,11,11)	[11,11]
D	(45,47,52,55)	$[45+2\alpha, 55-3\alpha]$	(12,15,19,22)	$[12+3\alpha, 22-3\alpha]$

The lower and upper bounds of (α,β)-cut of $\tilde{\theta}_A$ is calculated as follows:

$$
\begin{aligned}
\theta_A^{l(\alpha,\beta)} = \min \quad & \theta \\
s.t \quad & (14-2\alpha)\theta \geq (11+\beta)\lambda_1+30\lambda_2+40\lambda_3+(45+2\beta)\lambda_4+(14-2\beta)\mu_1+30\mu_2 \\
& \qquad +40\mu_3+(55-3\beta)\mu_4 \\
& 10 \leq 10\lambda_1+(16-2\beta)\lambda_2+11\lambda_3+(22-3\beta)\lambda_4+10\mu_1+(12+\beta)\mu_2+11\mu_3 \\
& \qquad +(12+3\beta)\mu_4 \\
& (14-2\alpha) \geq (11+\beta)\lambda_1+30\lambda_2+40\lambda_3+(45+2\beta)\lambda_4+(14-2\beta)\mu_1+30\mu_2 \\
& \qquad +40\mu_3+(55-3\beta)\mu_4 \\
& \lambda_j \geq 0, \quad j=1, \ldots, 4, \quad \mu_j \geq 0, \quad j=1, \ldots, 4
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\theta_A^{u(\alpha,\beta)} = \min \quad & \theta \\
s.t \quad & (11+\alpha)\theta \geq (14-2\beta)\lambda_1+30\lambda_2+40\lambda_3+(55-3\beta)\lambda_4+(11+\beta)\mu_1 \\
& \qquad +30\mu_2+40\mu_3+(45+2\beta)\mu_4 \\
& 10 \leq 10\lambda_1+(12+\beta)\lambda_2+11\lambda_3+(12+3\beta)\lambda_4+10\mu_1+(16-2\beta)\mu_2 \\
& \qquad +11\mu_3+(22-3\beta)\mu_4 \\
& (11+\alpha) \geq (14-2\beta)\lambda_1+30\lambda_2+40\lambda_3+(55-3\beta)\lambda_4+(11+\beta)\mu_1+30\mu_2 \\
& \qquad +40\mu_3+(45+2\beta)\mu_4 \\
& \lambda_j \geq 0, \quad j=1, \ldots, 4, \quad \mu_j \geq 0, \quad j=1, \ldots, 4
\end{aligned}
\tag{13}
$$

The lower and upper bounds of $(\alpha,\beta)-$ cuts of $\tilde{\theta}_B, \tilde{\theta}_C$ and $\tilde{\theta}_D$ can be solved similarly. The results are shown in Table 2 and table 3 for $\alpha=0.0,.1,.2,\ldots,1$, and $\beta=.3$ and $.7$. In DEA, we consider that $(x_0\ ,\ y_0) \in PPS$ and then we want to find the $\min\theta$, where $(\theta x_0\ ,\ y_0) \in PPS$. In Table 2, for $\alpha = 0.0,.1,.2.$ $(X_A^{l\alpha}, Y_A^{u\alpha}) \notin \tilde{P}^{.3}$. Hence, the model is infeasible (inf) and also in Table 3, for α=0.0,.1,.2,...,.6 $(X_A^{l\alpha}, Y_A^{u\alpha}) \notin \tilde{P}^{.7}$. Hence, the model is infeasible (inf) (see Propositions 1 and 2).

Table 2: $(\alpha\ , .3)$ - cuts of efficiency measures

α	$[\theta_A^l, \theta_A^r]$	$[\theta_B^l, \theta_B^r]$	$[\theta_C^l, \theta_C^r]$	$[\theta_D^l, \theta_D^r]$
0.0	*[.81,inf]	[.45,.60]	[.31,.31]	[.25,.55]
.2	*[.83,inf]	[.46,.59]	[.31,.31]	[.26,.53]
.3	[.84,1.0]	[.46,.58]	[.31,.31]	[.27,.52]
.4	[.86,.99]	[.47,.57]	[.31,.31]	[.28,.51]
.5	[.87,.98]	[.47,.56]	[.31,.31]	[.29,.49]
.6	[.88,.97]	[.47,.56]	[.31,.31]	[.29,.50]
.7	[.90,.97]	[.48,.55]	[.31,.31]	[.30,.48]
.8	[.91,.96]	[.48,.54]	[.31,.31]	[.31,.48]
.9	[.93,.95]	[.49,.53]	[.31,.31]	[.33,.46]
1.0	[.94,.94]	[.49,.53]	[.31,.31]	[.32,.47]

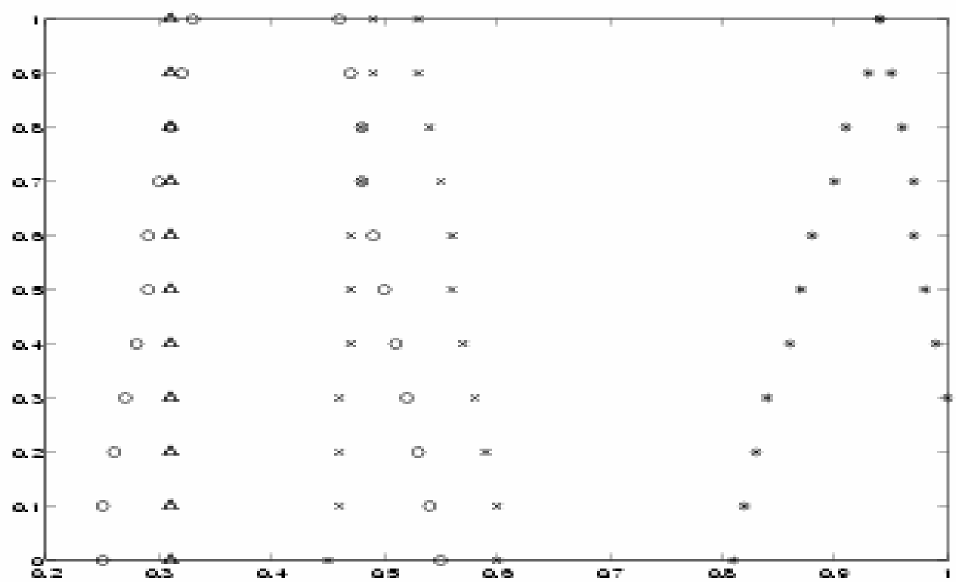

Figure 1: $(\tilde{\theta}_A\ ***), (\tilde{\theta}_B\ \times\times\times), (\tilde{\theta}_C\ \Delta\Delta\Delta)$ and $(\tilde{\theta}_D\ \circ\circ\circ)$

Table 3: $(\alpha, .7)$ - cuts of efficiency measures

α	$[\theta_A^l, \theta_A^r]$	$[\theta_B^l, \theta_B^r]$	$[\theta_C^l, \theta_C^r]$	$[\theta_D^l, \theta_D^r]$
0.0	*[.84,inf]	[.47,.62]	[.32,.32]	[.26,.57]
.1	*[.85,inf]	[.47,.62]	[.32,.32]	[.26,.56]
.2	*[.86,inf]	[.48,.61]	[.32,.32]	[.27,.55]
.3	*[.87,inf]	[.48,.60]	[.32,.32]	[.28,.54]
.4	*[.89,inf]	[.48,.59]	[.32,.32]	[.29,.53]
.5	*[.90,inf]	[.49,.58]	[.32,.32]	[.30,.52]
.6	*[.91,inf]	[.49,.58]	[.32,.32]	[.30,.51]
.7	[.93,1.0]	[.50,.57]	[.32,.32]	[.31,.50]
.8	[.94,.99]	[.50,.56]	[.32,.32]	[.32,.49]
.9	[.96,.98]	[.50,.55]	[.32,.32]	[.33,.48]
1.0	[.97,.97]	[.51,.55]	[.32,.32]	[.34,.47]

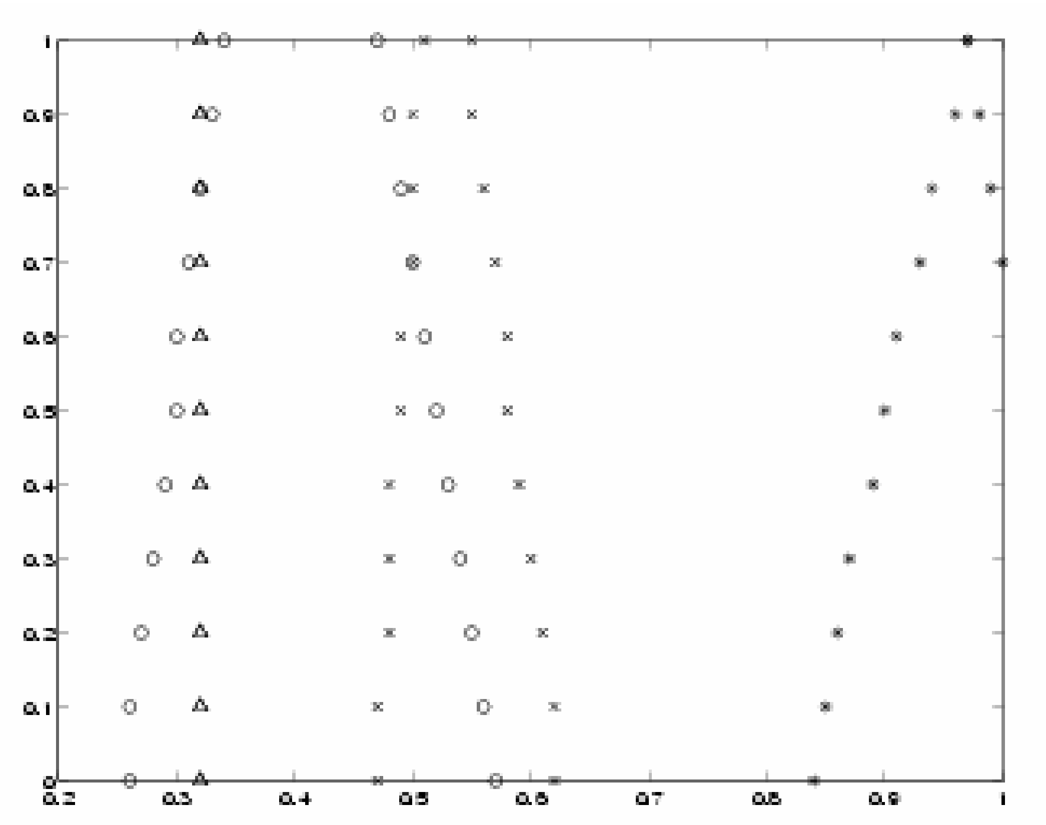

Figure 2: $(\tilde{\theta}_A\ ***), (\tilde{\theta}_B\ \times\times\times), (\tilde{\theta}_C\ \Delta\Delta\Delta)$ and $(\tilde{\theta}_D\ \circ\circ\circ)$

6. Conclusions

In the real world there are many problems which have fuzzy parameters. In this paper, we proposed input oriented *CCR* model, for evaluation of relative efficiency of *DMU*s in fuzzy production possibility set by considering initial concepts of *DEA*. This model can be extended to the other topics and models of *DEA*.

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