Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 4, Issue 2 (December 2009), pp. 279 – 289
(Previously, Vol. 4, No.2)

Applications and Applied Mathematics:
An International Journal
(AAM)

Explicit and Exact Solutions with Multiple Arbitrary Analytic Functions of Jimbo–Miwa Equation

Sheng Zhang, Ying-Na Sun, Jin-Mei Ba, Ling Dong
Department of Mathematics
Bohai University
Jinzhou 121000, PR China
zhshaeng@yahoo.com.cn

Received: May 22, 2009; Accepted: August 12, 2009

Abstract

In this paper, a generalized F-expansion method is used to construct exact solutions of the (3+1)-dimensional Jimbo–Miwa equation. As a result, many new and more general exact solutions are obtained including single and combined non-degenerate Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions, each of which contains six arbitrary analytic functions. It is shown that with the aid of symbolic computation the generalized F-expansion method may provide a straightforward and effective mathematical tool for solving nonlinear partial differential equations.

Keywords: Nonlinear partial differential equations; F-expansion method; Jacobi elliptic function solutions; Hyperbolic function solutions; Trigonometric function solutions

MSC (2000) No.: 35Q53, 35Q99

1. Introduction

Nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLPDEs will help one to understand these phenomena better. With the development of soliton theory, many effective methods for obtaining exact solutions of NLPDEs have been presented such as Hirota's bilinear method [Hirota (1971)], Bäcklund transformation [Miurs (1978)], Painlevé expansion [Weiss, Tabor and Carnevale (1983)], sine-cosine method [Yan (1996)], homogeneous balance method [Wang (1996)], homotopy perturbation method

[El-Shahed (2005), He (2005), Mohyud-Din and Noor (2008)], variational iteration method [He (1999), Noor and Mohyud-Din (2008a,b)], tanh-function method [Zayed et al. (2004), Abudsalam (2005), Zhang (2007a), Zhang and Xia (2006c)], algebraic method [Hu (2005), Yomba (2006), Zhang and Xia (2006b)], auxiliary equation method [Sirendaoreji (2003), Zhang and Xia (2007a,b)], and Exp-function method [Boz and Bekir (2008), He and Xu (2006), Yusufoglu (2008), Zhang (2008), Zhu (2007)].

Recently, a useful method called *F*-expansion method [Wang, et al. (2003), Wang and Zhou (2003), Zhou et al. (2003)] was proposed to construct periodic wave solutions of NLPDEs, which can be thought of as an over-all generalization of Jacobi elliptic function expansion method [Liu et al. (2001), Fu, et al. (2001), Parkes et al. (2002)]. The *F*-expansion method was later extended in different manners [Chen et al. (2005), Liu and Yang (2004), Ren and Zhang (2006), Wang and Li (2005), Zhang (2007c)]. More recently, we proposed a generalized *F*-expansion method [Zhang (2006, 2007b), Zhang and Xia (2006a)] to construct exact solutions which contain not only the results obtained by the methods of Chen, et al. (2005), Wang and Li (2005), Zhang (2007c) but also a series of new and more general exact solutions, in which the restriction on ξ as merely a linear function of $t, x_1, x_2, \cdots, x_m$ and the restriction on the coefficients being constants are all removed.

In the present paper, we further improve and develop our work made in [Zhang (2006), (2007b), Zhang and Xia (2006a)] for obtaining new and more general exact solutions of the (3+1)-dimensional Jimbo–Miwa (JM) equation [Senthilvelan (2001)]:

$$u_{xxxy} + 3(uu_y)_x + 3u_{xx}\partial_x^{-1}u_y + 3u_x u_y + 3u_{yt} - 3u_{zz} = 0, \tag{1}$$

the Painlevé property, Bäcklund transformation, soliton solutions and doubly periodic solutions were investigated by Lou (1996), Hong and Oh (2000), Zhang and Wu (2002).

The rest of this paper is organized as follows. In Section 2, we give the description of the generalized *F*-expansion method. In Section 3, we use this method to obtain more general exact solutions of the (3+1)-dimensional JM equation. In Section 4, some conclusions are given.

2. Description of the Generalized *F*-expansion Method

For a given NLPDE with independent variables $x = (t, x_1, x_2, \cdots, x_m)$ and dependent variable u

$$F(u, u_t, u_{x_1}, u_{x_2}, \cdots, u_{x_m}, u_{x_1 t}, u_{x_2 t} \cdots, u_{x_m t}, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, \cdots, u_{x_m x_m}, \cdots) = 0, \tag{2}$$

we seek its solutions in the more general form

$$u = a_0 + \sum_{i=1}^{n} \{a_i F^i(\xi) + b_i F^{-i}(\xi) + c_i F^{i-1}(\xi) F'(\xi) + d_i F^{-i}(\xi) F'(\xi)\}, \tag{3}$$

where $a_0 = a_0(x),\ a_i = a_i(x),\ b_i = b_i(x),\ c_i = c_i(x),\ d_i = d_i(x)\ (i = 1, 2, \cdots, n)$ and $\xi = \xi(x)$ are undetermined differentiable functions, $F(\xi)$ in (3) satisfies

$$F'^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \tag{4}$$

and, hence, holds for $F(\xi)$ and $F'(\xi)$

$$\begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi) \\ F'''(\xi) = (6PF^2(\xi) + Q)F'(\xi) \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi) \\ \qquad \vdots \end{cases} \tag{5}$$

where P, Q and R are all parameters, the prime denotes $\mathrm{d}/\mathrm{d}\xi$. Given different values of P, Q and $R,$ the different Jacobi elliptic function solutions $F(\xi)$ can be obtained from (4).

Some special solutions of (4) are listed in Table 1 including several new ones which weren't reported by Zhang (2006, 2007b), Zhang and Xia (2006a). To determine u explicitly, we take the following four steps:

***Step* 1.** Determine the integer n by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in (2).

***Step* 2.** Substitute (3) along with (4) and (5) into (2) and collect all the coefficients of $F'^i(\xi)F^j(\xi)\ (i = 0,1; j = 0, \pm 1, \pm 2, \cdots),$ then set each coefficient to zero to derive a set of over-determined partial differential equations for $a_0,\ a_i,\ b_i,\ c_i,\ d_i\ (i = 1, 2, \cdots, n)$ and $\xi.$

***Step* 3.** Solve the system of over-determined partial differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, d_i and ξ by use of *Mathematica*.

***Step* 4.** Select the appropriate P, Q, R and $F(\xi)$ from Table 1 and substitute them along with a_0, a_i, b_i, c_i, d_i and ξ into (3) to obtain non-degenerate Jacobi elliptic function solutions of (2) (see Table 2 for $F'(\xi)$), from which hyperbolic function solutions and trigonometric function solutions can be obtained in the limit cases when $m \to 1$ and $m \to 0$ (see Tables 3 and 4).

Table 1. Relations between values of P, Q, R and corresponding $F(\xi)$ of (4)

P	Q	R	$F(\xi)$
m^2	$-(1+m)^2$	1	$\mathrm{sn}\xi,\quad \mathrm{cd}\xi = \dfrac{\mathrm{cn}\xi}{\mathrm{dn}\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{cn}\xi$
-1	$2-m^2$	m^2-1	$\mathrm{dn}\xi$
1	$-(1+m)^2$	m^2	$\mathrm{ns}\xi = \mathrm{sn}^{-1}\xi,\quad \mathrm{dc}\xi = \dfrac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{nc}\xi = \mathrm{cn}^{-1}\xi$
m^2-1	$2-m^2$	-1	$\mathrm{nd}\xi = \mathrm{dn}^{-1}\xi$

$1-m^2$	$2-m^2$	1	$\mathrm{sc}\xi=\dfrac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{sd}\xi=\dfrac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\xi=\dfrac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\xi=\dfrac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\dfrac{1}{4}$	$\dfrac{1-2m^2}{2}$	$\dfrac{1}{4}$	$\mathrm{ns}\xi\pm\mathrm{cs}\xi,\quad \dfrac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{dn}\xi}$
$\dfrac{1-m^2}{4}$	$\dfrac{1+m^2}{2}$	$\dfrac{1-m^2}{4}$	$\mathrm{nc}\xi\pm\mathrm{sc}\xi$
$\dfrac{1}{4}$	$\dfrac{m^2-2}{2}$	$\dfrac{m^2}{4}$	$\mathrm{ns}\xi\pm\mathrm{ds}\xi$
$\dfrac{m^2}{4}$	$\dfrac{m^2-2}{2}$	$\dfrac{m^2}{4}$	$\mathrm{sn}\xi\pm i\mathrm{cn}\xi,\quad \dfrac{\mathrm{dn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{cn}\xi}$
$-\dfrac{1}{4}$	$\dfrac{1+m^2}{2}$	$-\dfrac{(1-m^2)^2}{4}$	$m\mathrm{cn}\xi\pm\mathrm{dn}\xi$
$\dfrac{1}{4}$	$\dfrac{1-2m^2}{2}$	$\dfrac{1}{4}$	$m\mathrm{sn}\xi\pm i\mathrm{dn}\xi,\quad \dfrac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi}$
$\dfrac{1}{4m^2}$	$\dfrac{1-2m^2}{2}$	$\dfrac{m^2}{4}$	$\dfrac{\mathrm{dn}\xi}{\sqrt{\dfrac{1-m^2}{m^2}}\pm\mathrm{cn}\xi}$
$\dfrac{m^2}{4}$	$\dfrac{m^2-2}{2}$	$\dfrac{1}{4}$	$\dfrac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi}$
$\dfrac{1-m^2}{4}$	$\dfrac{1+m^2}{2}$	$\dfrac{m^2-1}{4}$	$\dfrac{\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}$
$\dfrac{1-m^2}{4}$	$\dfrac{1+m^2}{2}$	$\dfrac{1-m^2}{4}$	$\dfrac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$
$\dfrac{(1-m^2)^2}{4}$	$\dfrac{1+m^2}{2}$	$\dfrac{1}{4}$	$\dfrac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$
$\dfrac{m^4}{4}$	$\dfrac{2-m^2}{2}$	$\dfrac{1}{4}$	$\dfrac{\mathrm{cn}\xi}{\sqrt{1-m^2}\pm\mathrm{dn}\xi}$

Table 2. Derivatives of Jacobi Elliptic Functions

$\mathrm{sn}'\xi=\mathrm{cn}\xi\mathrm{dn}\xi$	$\mathrm{cd}'\xi=-(1-m^2)\mathrm{sd}\xi\mathrm{nd}\xi$	$\mathrm{cn}'\xi=-\mathrm{sn}\xi\mathrm{dn}\xi$
$\mathrm{dn}'\xi=-m^2\mathrm{sn}\xi\mathrm{cn}\xi$	$\mathrm{ns}'\xi=-\mathrm{cs}\xi\mathrm{ds}\xi$	$\mathrm{dc}'\xi=(1-m^2)\mathrm{nc}\xi\mathrm{sc}\xi$
$\mathrm{nc}'\xi=\mathrm{sc}\xi\mathrm{dc}\xi$	$\mathrm{nd}'\xi=m^2\mathrm{cd}\xi\mathrm{sd}\xi$	$\mathrm{sc}'\xi=\mathrm{dc}\xi\mathrm{nc}\xi$
$\mathrm{cs}'\xi=-\mathrm{ns}\xi\mathrm{ds}\xi$	$\mathrm{ds}'\xi=-\mathrm{cs}\xi\mathrm{ns}\xi$	$\mathrm{sd}'\xi=\mathrm{nd}\xi\mathrm{cd}\xi$

Table 3. Jacobi Elliptic Functions Degenerate into Hyperbolic Functions when $m \to 1$

$\text{sn}\xi \to \tanh\xi$	$\text{cn}\xi \to \text{sech}\xi$	$\text{dn}\xi \to \text{sech}\xi$
$\text{sc}\xi \to \sinh\xi$	$\text{sd}\xi \to \sinh\xi$	$\text{cd}\xi \to 1$
$\text{ns}\xi \to \coth\xi$	$\text{nc}\xi \to \cosh\xi$	$\text{nd}\xi \to \cosh\xi$
$\text{cs}\xi \to \text{csch}\xi$	$\text{ds}\xi \to \text{csch}\xi$	$\text{dc}\xi \to 1$

Table 4. Jacobi Elliptic Functions Degenerate into Trigonometric Functions when $m \to 0$

$\text{sn}\xi \to \sin\xi$	$\text{cn}\xi \to \cos\xi$	$\text{dn}\xi \to 1$
$\text{sc}\xi \to \tan\xi$	$\text{sd}\xi \to \sin\xi$	$\text{cd}\xi \to \cos\xi$
$\text{ns}\xi \to \csc\xi$	$\text{nc}\xi \to \sec\xi$	$\text{nd}\xi \to 1$
$\text{cs}\xi \to \cot\xi$	$\text{ds}\xi \to \csc\xi$	$\text{dc}\xi \to \sec\xi$

Remark 1. The solution hypothesis (3), based on the most existing work, is a trail solution of NLPDEs. It can be easily found that (3) is more general than those introduced in literature. To be more precise, if $b_i = c_i = d_i = 0$, a_0 and a_i are constants, and ξ is merely a linear function of x and t, then (3) becomes those constructed by Wang, et al. (2003), Wang and Zhou (2003), Zhou et al. (2003). If $c_i = d_i = 0$, a_0, a_i and b_i are constants, and ξ is merely a linear function of x and t, then (3) reduces to that used by Wang and Li (2005). If $c_i = d_i = 0$, then (3) changes into the one employed by Chen et al. (2005). If $d_i = 0$, then (3) gives the general one proposed by Zhang (2007c).

3. Exact Solutions of the JM Equation

In order to obtain exact solutions of (1), we suppose $u_y = v_x$ and set the integral constant as zero, then (1) becomes

$$u_{xxxy} + 6u_x u_y + 3uv_{xx} + 3u_{xx}v + 3u_{yt} - 3u_{zz} = 0, \tag{6}$$

$$u_y - v_x = 0. \tag{7}$$

According to Step 1, we get $n = 2$ for u and v. To search for explicit and exact solutions, we assume that (6) and (7) have the following formal solutions:

$$u = a_0 + a_1F(\xi) + a_2F^2(\xi) + b_1F^{-1}(\xi) + b_2F^{-2}(\xi) + c_1F'(\xi) + c_2F'(\xi)F(\xi) + d_1F'(\xi)F^{-1}(\xi) + d_2F'(\xi)F^{-2}(\xi), \tag{8}$$

$$v = A_0 + A_1F(\xi) + A_2F^2(\xi) + B_1F^{-1}(\xi) + B_2F^{-2}(\xi) + C_1F'(\xi) + C_2F'(\xi)F(\xi) + D_1F'(\xi)F^{-1}(\xi) + D_2F'(\xi)F^{-2}(\xi), \tag{9}$$

where $a_0 = a_0(y,z,t)$, $a_i = a_i(y,z,t)$, $b_i = b_i(y,z,t)$, $c_i = c_i(y,z,t)$, $d_i = d_i(y,z,t)$, $A_0 = A_0(y,z,t)$, $A_i = A_i(y,z,t)$, $B_i = B_i(y,z,t)$, $C_i = C_i(y,z,t)$, $D_i = D_i(y,z,t)$ $(i=1,2)$, $\eta = \eta(y,z,t)$, $\xi = kx + \eta$, k is a nonzero constant.

With the aid of *Mathematica*, substituting (8) and (9) along with (4) and (5) into (6) and (7), the left-hand sides of (6) and (7) are converted into two polynomials of $F'^i(\xi)F^j(\xi)$ $(i=0,1; j=0,\pm1,\pm2,\cdots)$, then setting each coefficient to zero, we get a set of over-determined partial differential equations for a_0, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, A_0, A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 and η. Solving the set of over-determined partial differential equations by use of *Mathematica*, we get the following results:

***Case* 1.**

$$a_0 = zg_1(t) + g_2(t),\quad a_1 = 0,\quad a_2 = -k^2P,\quad b_1 = 0,\quad b_2 = -k^2R,$$
$$c_1 = \pm k^2\sqrt{P},\quad c_2 = 0,\quad d_1 = 0,\quad d_2 = \pm k^2\sqrt{R},$$
$$A_0 = \frac{3\eta_z^2 + k^3(\pm 6\sqrt{PR} - Q)\eta_y - 3k(zg_1(t) + g_2(t))\eta_y - 3\eta_y\eta_t}{3k^2},\quad A_1 = 0,$$
$$A_2 = -kP\eta_y,\quad B_1 = 0,\quad B_2 = -kR\eta_y,\quad C_1 = \pm k\sqrt{P}\eta_y,\quad C_2 = 0,\quad D_1 = 0,$$
$$D_2 = \pm k\sqrt{R}\eta_y,\quad \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t),$$

where $f_1(y)$, $f_2(t)$, $f_3(y)$, $f_4(t)$, $g_1(t)$ and $g_2(t)$ are arbitrary analytic functions of the indicated variables, l is an arbitrary constant.

***Case* 2.**

$$a_0 = zg_1(t) + g_2(t),\quad a_1 = 0,\quad a_2 = -k^2P,\quad b_1 = 0,\quad b_2 = 0,\quad c_1 = \pm k^2\sqrt{P},$$
$$c_2 = 0,\quad d_1 = 0,\quad d_2 = 0,\quad A_0 = \frac{3\eta_z^2 - k^3Q\eta_y - 3k(zg_1(t) + g_2(t))\eta_y - 3\eta_y\eta_t}{3k^2},$$
$$A_1 = 0,\quad A_2 = -kP\eta_y,\quad B_1 = 0,\quad B_2 = 0,\quad C_1 = \pm k\sqrt{P}\eta_y,\quad C_2 = 0,\quad D_1 = 0,$$
$$D_2 = 0,\quad \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t),$$

where $f_1(y)$, $f_2(t)$, $f_3(y)$, $f_4(t)$, $g_1(t)$ and $g_2(t)$ are arbitrary analytic functions of the indicated variables, l is an arbitrary constant.

***Case* 3.**

$$a_0 = zg_1(t) + g_2(t),\quad a_1 = 0,\quad a_2 = -2k^2P,\quad b_1 = 0,\quad b_2 = -2k^2R,\quad c_1 = 0,$$
$$c_2 = 0,\quad d_1 = 0,\quad d_2 = 0,\quad A_0 = \frac{3\eta_z^2 - 4k^3Q\eta_y - 3k(zg_1(t) + g_2(t))\eta_y - 3\eta_y\eta_t}{3k^2},$$
$$A_1 = 0,\quad A_2 = -kP\eta_y,\quad B_1 = 0,\quad B_2 = -kR\eta_y,\quad C_1 = 0,\quad C_2 = 0,\quad D_1 = 0,$$
$$D_2 = 0,\quad \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t),$$

where $f_1(y),\ f_2(t),\ f_3(y),\ f_4(t),\ g_1(t)$ and $g_2(t)$ are arbitrary analytic functions of the indicated variables, l is an arbitrary constant.

***Case* 4.**

$$a_0 = zg_1(t) + g_2(t),\quad a_1 = 0,\quad a_2 = -2k^2P,\quad b_1 = 0,\quad b_2 = 0,\quad c_1 = 0,$$
$$c_2 = 0,\quad d_1 = 0,\quad d_2 = 0,\quad A_0 = \frac{3\eta_z^2 - 4k^3Q\eta_y - 3k(zg_1(t) + g_2(t))\eta_y - 3\eta_y\eta_t}{3k^2},$$
$$A_1 = 0,\quad A_2 = -kP\eta_y,\quad B_1 = 0,\quad B_2 = 0,\quad C_1 = 0,\quad C_2 = 0,\quad D_1 = 0,$$
$$D_2 = 0,\quad \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t),$$

where $f_1(y),\ f_2(t),\ f_3(y),\ f_4(t),\ g_1(t)$ and $g_2(t)$ are arbitrary analytic functions of the indicated variables, l is an arbitrary constant.

Selecting $F(\xi) = \mathrm{ns}\xi,\ P = 1,\ Q = -(1+m^2),\ R = m^2$ from Table 1, using Case 1 and Table 2, we obtain combined non-degenerate Jacobi elliptic function solutions:

$$u = zg_1(t) + g_2(t) - k^2\mathrm{ns}^2\xi - k^2m^2\mathrm{sn}^2\xi \mp k^2\mathrm{cs}\xi\mathrm{ds}\xi \mp k^2m\mathrm{cn}\xi\mathrm{dn}\xi, \tag{10}$$

$$v = \frac{3\eta_z^2 + k^3(\pm 6m + 1 + m^2)\eta_y - 3k(zg_1(t) + g_2(t))\eta_y - 3\eta_y\eta_t}{3k^2} - k\eta_y\mathrm{ns}^2\xi$$
$$-km^2\eta_y\mathrm{sn}^2\xi \mp k\eta_y\mathrm{cs}\xi\mathrm{ds}\xi \mp km\eta_y\mathrm{cn}\xi\mathrm{dn}\xi, \tag{11}$$

where $\xi = kx + \eta,\ \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t).$

Selecting again $F(\xi) = \mathrm{ns}\xi \pm \mathrm{cs}\xi,\ P = 1/4,\ Q = -(1-2m^2)/2,\ R = 1/4$ from Table 1, using Case 1 and Table 2, we obtain combined non-degenerate Jacobi elliptic function solutions:

$$u = zg_1(t) + g_2(t) - \frac{k^2}{4}(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2 - \frac{k^2}{4}\frac{1}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2}$$
$$\mp \frac{k^2}{2}(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi) \mp \frac{k^2}{2}\frac{\pm\mathrm{ds}\xi}{\mathrm{ns}\xi \pm \mathrm{cs}\xi}, \tag{12}$$

$$v = \frac{6\eta_z^2 + k^3(\pm 3 - 1 + 2m^2)\eta_y - 6k(zg_1(t) + g_2(t))\eta_y - 6\eta_y\eta_t}{6k^2} - \frac{k}{4}\eta_y(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2$$
$$-\frac{k}{4}\eta_y\frac{1}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2} \mp \frac{k}{2}\eta_y(\mathrm{cs}\xi\mathrm{ds}\xi \pm \mathrm{ns}\xi\mathrm{ds}\xi) \mp \frac{k}{2}\eta_y\frac{\mp\mathrm{ds}\xi}{\mathrm{ns}\xi \pm \mathrm{cs}\xi}, \tag{13}$$

where $\xi = kx + \eta,\ \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t).$

In the limit case when $m \to 1$, from (12) and (13) we get hyperbolic function solutions:

$$\begin{aligned} u = zg_1(t) + g_2(t) - \frac{k^2}{4}(\coth\xi \pm \mathrm{csch}\xi)^2 - \frac{k^2}{4}\frac{1}{(\coth\xi \pm \mathrm{csch}\xi)^2} \\ \mp \frac{k^2}{2}(\mathrm{csch}^2\xi \pm \coth\xi\,\mathrm{csch}\xi) \mp \frac{k^2}{2}\frac{\mp\mathrm{csch}\xi}{\coth\xi \pm \mathrm{csch}\xi}, \end{aligned} \tag{14}$$

$$\begin{aligned} v = \frac{6\eta_z^2 + k^3(\pm 3 + 1)\eta_y - 6k(zg_1(t) + g_2(t))\eta_y - 6\eta_y\eta_t}{6k^2} - \frac{k}{4}\eta_y(\coth\xi \pm \mathrm{csch}\xi)^2 \\ - \frac{k}{4}\eta_y\frac{1}{(\coth\xi \pm \mathrm{csch}\xi)^2} \mp \frac{k}{2}\eta_y(\mathrm{csch}^2\xi \pm \coth\xi\,\mathrm{csch}\xi) \mp \frac{k}{2}\eta_y\frac{\mp\mathrm{csch}\xi}{\coth\xi \pm \mathrm{csch}\xi}, \end{aligned} \tag{15}$$

where $\xi = kx + \eta,\ \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t).$

In the limit case when $m \to 0$, from (12) and (13) we get trigonometric function solutions:

$$\begin{aligned} u = zg_1(t) + g_2(t) - \frac{k^2}{4}(\csc\xi \pm \cot\xi)^2 - \frac{k^2}{4}\frac{1}{(\csc\xi \pm \cot\xi)^2} \\ \mp \frac{k^2}{2}(\cot\xi\csc\xi \pm \csc^2\xi) \mp \frac{k^2}{2}\frac{\mp\csc\xi}{\csc\xi \pm \cot\xi}, \end{aligned} \tag{16}$$

$$\begin{aligned} v = \frac{6\eta_z^2 + k^3(\pm 3 - 1)\eta_y - 6k(zg_1(t) + g_2(t))\eta_y - 6\eta_y\eta_t}{6k^2} - \frac{k}{4}\eta_y(\csc\xi \pm \cot\xi)^2 \\ - \frac{k}{4}\eta_y\frac{1}{(\csc\xi \pm \cot\xi)^2} \mp \frac{k}{2}\eta_y(\cot\xi\csc\xi \pm \csc^2\xi) \mp \frac{k}{2}\eta_y\frac{\mp\csc\xi}{\csc\xi \pm \cot\xi}, \end{aligned} \tag{17}$$

where $\xi = kx + \eta,\ \eta = l(z^3 + 6yzt + z^2 + 2yt) + zf_1(y) + zf_2(t) + f_3(y) + f_4(t).$

To the best of our knowledge, the solutions obtained above have not been reported in the literature. From Cases 1–4, we can also obtain other Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions, here we omit them for simplicity.

4. Conclusions

In this paper, we have successfully constructed new and more general exact solutions with six arbitrary analytic functions of the (3+1)-dimensional JM equation including single and combined non-degenerate Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions. These solutions contain six arbitrary analytic functions which can make us discuss the behaviors of solutions and also provide us with enough freedom to construct solutions that may be related to real physical problems. It may be important to explain some physical phenomena.

Acknowledgments

The authors would like to thank the referees for their valuable suggestions and comments. This work was supported by the Natural Science Foundation of Educational Committee of Liaoning Province of China under Grant No. 20060022.

REFERENCES

Abudsalam, H. A. (2005). On an improved complex tanh-function method. *Int. J. Nonlinear Sci. Numer. Simul.* **6**, 99–106.

Boz, A. and Bekir, A. (2008). Application of Exp-function method for (3+1)-dimensional nonlinear evolution equations. *Comput. Math. Appl.* **56**, 451–1456.

Chen, J., He, H. S. and Yang, K. Q. (2005). A generalized *F*-expansion method and its application in high-dimensional nonlinear evolution equation. *Commun. Theor. Phys. (Beijing, China)* **44**, 307–310.

El-Shahed, M. (2005). Application of He's homotopy perturbation method to Volterra's integro-differential equation. *Int. J. Nonlinear Sci. Numer. Simul.* **6**, 163–168.

Fu, Z. T., Liu, S. K., Liu, S. D. and Zhao, Q. (2001). New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations. *Phys. Lett. A* **290**, 72–76.

He, J. H. (1999). Variational iteration method—a kind of nonlinear analytical technique: some examples. *Int. J. Nonlinear Mech.* **34**, 699–708.

He, J. H. (2005). Homotopy perturbation method for bifurcation of nonlinear problems. *Int. J. Nonlinear Sci. Numer. Simul.* **6**, 207–208.

He, J. H. and Wu, X. H (2006). Exp-function method for nonlinear wave equations. *Chaos, Solitons & Fractals* **30**, 700–708.

Hirota, R. (1971). Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27**, 1192–1194.

Hong, W. and Oh, K. S. (2000). New solitonic solutions to a (3+1)-dimensional Jimbo–Miwa equation. *Comput. Math. Appl.* **39**, 29–31.

Hu, J. Q. (2005). An algebraic method exactly solving two high-dimensional nonlinear evolution equations. *Chaos, Solitons & Fractals* **23**, 391–398.

Liu, J. B. and Yang, K. Q. (2004). The extended *F*-expansion method and exact solutions of nonlinear PDEs. *Chaos, Solitons & Fractals* **22**, 111–121.

Liu, S. K., Fu, Z. T., Liu, S. D. and Zhao, Q. (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Phys. Lett. A* **289**, 69–74.

Lou, S. Y. (1996). Dromion-like structures in a (3+1)-dimensional KdV-type equation, *J. Phys. A* **29**, 5989–6002.

Miurs, M.R. (1978). *Bäcklund Transformation*, Springer, Berlin.

Mohyud-Din, S. T. and Noor, M. A. (2008). Homotopy perturbation method and padé approximants for solving Flierl–Petviashivili equation. *Appl. Appl. Math.: An Int. J.* **3**, 224–234.

Noor, M. A. and Mohyud-Din, S. T. (2008a). A reliable approach for higher-order integro-differential equations. *Appl. Appl. Math.: An Int. J.* **3**, 188–199.

Noor, M. A. and Mohyud-Din, S. T. (2008b). Solving higher dimensional initial boundary value problems by variational iteration decomposition method. *Appl. Appl. Math.: An Int. J.* **3**, 254–266.

Parkes, E. J., Duffy, B. R. and Abbott, P. C. (2002). The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations. *Phys. Lett. A* **295**, 280–286.

Ren, Y. J. and Zhang, H. Q. (2006). A generalized *F*-expansion method to find abundant families of Jacobi elliptic function solutions of the (2+1)-dimensional Nizhnik–Novikov–Veselov equation. *Chaos, Solitons & Fractals* **27**, 959–979.

Senthilvelan, M. (2001). On the extended applications of homogeneous balance method, *Appl. Math. Comput.* **123**, 381–388.

Sirendaoreji, Sun, J. (2003). Auxiliary equation method for solving nonlinear partial differential equations. *Phys. Lett. A* **309**, 387–396.

Wang, D. S. and Zhang, H. Q. (2005). Further improved *F*-expansion method and new exact solutions of Konopelchenko–Dubrovsky equation. *Chaos, Solitons & Fractals* **25**, 601–610.

Wang, M. L. (1996). Exact solution for a compound KdV–Burgers equations. *Phys. Lett. A* **213**, 279–287.

Wang, M. L. and Li, X. Z. (2005). Applications of *F*-expansion to periodic wave solutions for a new Hamiltonian amplitude equation. *Chaos, Solitons & Fractals* **24**, 1257–1268.

Wang, M. L., Wang, Y. M. and Zhang, J. L. (2003). The periodic wave solutions for two systems of nonlinear wave equations. *Chin. Phys.* **12**, 1341–1348.

Wang, M. L. and Zhou, Y. B. (2003). The periodic wave solutions for the Klein–Gordon–Schrödinger equations. *Phys. Lett. A* **318**, 84–92.

Weiss, J., Tabor, M. and Carnevale, G. (1983). The Painlevé property for partial differential equations. *J. Math. Phys.* **24**, 522–526.

Yan, C. T. (1996). A simple transformation for nonlinear waves, *Phys. Lett. A* **224**, 77–84.

Yomba, E. (2006). The modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer–Kaup–Kupershmidt equation. *Chaos, Solitons & Fractals* **27**, 187–196.

Yusufoglu, E. (2008). New solitary solutions for the MBBM equations using Exp-function method. *Phys. Lett. A* **372**, 442–446.

Zayed, E. M. E., Zedan, H. A. and Gepreel, K. A. (2004). Group analysis and modified extended tanh-function to find the invariant solutions and soliton solutions for nonlinear Euler equations. *Int. J. Nonlinear Sci. Numer. Simul.* **5**, 221–234.

Zhang, J. F. and Wu, F. M. (2002). Bäcklund transformation and multiple soliton solutions for the (3+1)-dimensional Jimbo–Miwa equation. *Chin. Phys.* **11**, 425–428.

Zhang, S. (2006). New exact solutions of the KdV–Burgers–Kuramoto equation. *Phys. Lett. A* **358**, 414–420.

Zhang, S. (2007a). Symbolic computation and new families of exact non-travelling wave solutions of (2+1)-dimensional Konopelchenko–Dubrovsky equations. *Chaos, Solitons & Fractals* **31**, 951–959.

Zhang, S. (2007b). A generalized *F*-expansion method with symbolic computation exactly solving Broer-Kaup equations. *Appl. Math. Comput.* **189**, 836–843.

Zhang, S. (2007c). Further improved *F*-expansion method and new exact solutions of Kadomstev–Petviashvili equation. *Chaos, Solitons & Fractals* **32**, 1375–1383.

Zhang, S. (2008). Exp-function method exactly solving a KdV equation with forcing term. *Appl. Math. Comput.* **197**, 128–134.

Zhang, S. and Xia, T. C. (2006a). A generalized *F*-expansion method and new exact solutions of Konopelchenko-Dubrovsky equations. *Appl. Math. Comput.* **183**, 1190–1200.

Zhang, S. and Xia, T. C. (2006b). Further improved extended Fan sub-equation method and new exact solutions of the (2+1)-dimensional Broer–Kaup–Kupershmidt equations. *Appl. Math. Comput.* **182**, 1651–1660.

Zhang, S. and Xia, T. C. (2006c). Symbolic computation and new families of exact non-travelling wave solutions of (3+1)-dimensional Kadomstev–Petviashvili equation. *Appl. Math. Comput.* **181**, 319–331.

Zhang, S. and Xia, T. C. (2007a). A generalized auxiliary equation method and its application to (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations. *J. Phys. A: Math. Theor.* **40**, 227–248.

Zhang, S. and Xia T. C. (2007b). A generalized new auxiliary equation method and its applications to nonlinear partial differential equations. *Phys. Lett. A* **363**, 356–360.

Zhou, Y. B., Wang, M. L. and Wang, Y. M. (2003). Periodic wave solutions to a coupled KdV equations with variable coefficients. *Phys. Lett. A* **308**, 31–36.

Zhu, S. D. (2007). Exp-function method for the hybrid-lattice system. *Int. J. Nonlinear Sci. Numer. Simul.* **8**, 461–464.