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Establishment of a Chebyshev-dependent Inhomogeneous Second Order Differential Equation for the Applied Physics-related Boubaker-Turki Polynomials

Micahel Dada
O. Bamidele Awojoyogbe
Department of Physics
Federal University of Technology
Minna, Niger-State, Nigeria

Maximilian Hasler
Laboratoire A.O.C.
Université Antilles-Guyane, D.S.I., B.P. 7209
97275 Schoelcher cedex. MARTINIQUE, FRANCE

Karem B. Ben Mahmoud
Almas, Esstt, 1135 Naassan, Tunisia

Amine Bannour*
FMD Cluj-Napoca, ROUMANIA
**Corresponding author*

Abstract

This paper proposes Chebyshev-dependent inhomogeneous second order differential equation for the m-Boubaker polynomials (or Boubaker-Turki polynomials). This differential equation is also presented as a guide to applied physics studies. A concrete example is given through an attempt to solve the Bloch NMR flow equation inside blood vessels.

Keywords: Polynomial expansion, Differential equation, Applied physics

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1. Introduction

The Boubaker polynomials are the components of a special function which was established while studying an applied physics model by Chaouachi et al. (2007). This special function and its usefulness have been discussed in several studies by Boubaker (2007, 2008). The modified Boubaker polynomials (or Boubaker-Turki polynomials) is an enhanced form of these polynomials. Oppositely to the earlier defined polynomials, the Boubaker-Turki polynomials have a characteristic differential equation established by Labiadh et al. (2008) as well as an original ordinary generating function demonstrated by Awojoyogbe et al. (2009).

In this paper we tried to establish, for the first time, a Chebyshev-dependent differential equation for the Boubaker-Turki polynomials as a guide to particular applied physics studies.

2. Historic preview

2.1. The Boubaker Polynomials

According to their first definition in an attempt to solve the heat equation, the Boubaker polynomials have the explicit expression:

$$B_n(X) = \sum_{p=0}^{\xi(n)} \left[\frac{(n-4p)}{(n-p)} C_{n-p}^{p} \right].(-1)^p.X^{n-2p} \quad , \tag{1}$$

where the symbol $\lfloor \ \rfloor$ designates the *floor* function. Their coefficients could be defined through a recursive formula (2) that yields the first few polynomials (3):

$$\begin{cases} B_n(X) = \sum_{j=0}^{\xi(n)} \left[b_{n,j} X^{n-2j} \right]; \ \xi(n) = \dfrac{2n + ((-1)^n - 1)}{4}; \\ b_{n,0} = 1; \qquad b_{n,1} = -(n-4); \\ b_{n,j+1} = \dfrac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \dfrac{(n-4j-4)}{(n-4j)} \times b_{n,j}; \\ b_{n,\frac{2n+((-1)^n-1)}{4}} = \begin{cases} (-1)^{\frac{n}{2}} \times 2, & \text{if n even,} \\ (-1)^{\frac{n+1}{2}} (n-2), & \text{if n odd.} \end{cases} \end{cases} \tag{2}$$

$$\begin{cases} B_0(\mathrm{X}) = 1; \\ B_1(\mathrm{X}) = X; \\ B_2(\mathrm{X}) = X^2 + 2; \\ B_3(\mathrm{X}) = X^3 + X; \\ B_4(\mathrm{X}) = X^4 - 2; \\ B_5(\mathrm{X}) = X^5 - X^3 - 3X; \\ B_6(\mathrm{X}) = X^6 - 2X^4 - 3X^2 + 2; \\ B_7(\mathrm{X}) = X^7 - 3X^5 - 2X^3 + 5X; \\ B_8(\mathrm{X}) = X^8 - 4X^6 + 8X^2 - 2; \\ B_9(X) = X^9 - 5X^7 + 3X^5 + 10X^3 - 7X; \\ \ldots \\ B_m(X) = X.B_{m-1}(X) - B_{m-2}(X), \qquad \text{for } m > 2. \end{cases} \tag{3}$$

2.2. The Modified Boubaker Polynomials

The Modified Boubaker polynomials have been proposed through a specialized study. They are defined by (4):

$$\tilde{B}_n(X) = 2^n X^n - 2^{n-2}(n-4)X^{n-2} + \sum_{p=2}^{\xi(n)} \left[\frac{(n-4p)}{p!} \prod_{j=p+1}^{2p-1} (n-j) \right] \cdot 2^{n-2p} (-1)^p X^{n-2p} \ , \ \xi(n) = \frac{2n + ((-1)^n - 1)}{4}. \tag{4}$$

They are solutions to a second order characteristic equation (5):

$$(X^2 - 1)(3nX^2 + n - 2)y'' + 3X(nX^2 + 3n - 2)y' - n(3X^2n^2 + n^2 - 6n + 8)y = 0 . \tag{5}$$

The modified Boubaker polynomials have a recursive monomial definition expressed by equation (6):

$$\begin{cases} \tilde{B}_n(X) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left[b_{n,j} X^{n-2j} \right]; \\ \tilde{b}_{n,0} = 2^n; \qquad \tilde{b}_{n,1} = -2^{n-2}(n-4); \\ \tilde{b}_{n,j+1} = \frac{(n-2j)(n-2j-1)}{(j+1)(n-j-1)} \times \frac{(n-4j-4)}{(n-4j)} \times \tilde{b}_{n,j}; \\ \tilde{b}_{n,2\left\lfloor \frac{n}{2} \right\rfloor} = \begin{cases} (-1)^{\frac{n}{2}} \times 2, & \text{if } n \text{ even}, \\ 2(-1)^{\frac{n+1}{2}} \times 2(n-2), & \text{if } n \text{ odd}. \end{cases} \end{cases} \tag{6}$$

This definition allowed an establishment of a quasi-polynomial expression (7) of the *m*-Boubaker polynomials:

$$\tilde{B}_n(X) = \left\langle X+\sqrt{X^2-1}\right\rangle^n \left[8X^2-3-8X\sqrt{X^2-1}\right] + \left\langle X-\sqrt{X^2-1}\right\rangle^n \left[8X^2-3+8X\sqrt{X^2-1}\right]. \tag{7}$$

3. The Ordinary Generating Function of the Boubaker-Turki Polynomials

H. Labiadh et al. (2007) succeeded to establish an ordinary generating to the Boubaker-Turki polynomials that verifies (8):

$$f_{\tilde{B}}(X,t) = \sum_{n=0}^{\infty} \tilde{B}_n(X)\cdot t^n. \tag{8}$$

This ordinary generating function is expressed by (9):

$$f_{\tilde{B}}(X,t) = \left(1+3t^2\right)\left(1-t\left(2X-t\right)\right)^{-1}. \tag{9}$$

4. A Chebyshev-dependent Second Order Differential Equation for the Boubaker-Turki Polynomials

According to the recent results, the Boubaker-Turki polynomials can be expressed by (10):

$$\tilde{B}_n(X) = \frac{4X}{n}\frac{dT_n(X)}{dX} - 2T_n(X), \tag{10}$$

where $T_n(X)$, for $n > 2$, are the Chebyshev first order polynomials.

Equation (10) gives (11)

$$\frac{d\tilde{B}_n(X)}{dX} = \frac{4X}{n}\frac{d^2T_n(X)}{dX^2} + \left(\frac{4}{n}-2\right)\frac{dT_n(X)}{dX} \tag{11}$$

and (12)

$$\frac{d^2\tilde{B}_n(X)}{dX^2} = \frac{4X}{n}\frac{d^3T_n(X)}{dX^3} + \left(\frac{8}{n}-2\right)\frac{d^2T_n(X)}{dX^2}, \tag{12}$$

using the third order differential equation (13)

$$(1-x^2)\frac{d^2T_n(X)}{dX^2} - 3X\frac{d^3T_n(X)}{dX^3} - \left(n^2-1\right)\frac{dT_n(X)}{dX}, \tag{13}$$

we obtain the second order differential equation (14)

$$4X(1-X^2)y''+P(X,n)y'+Q(X,n)y=2Q(X,n)T_n(X), \tag{14}$$

with

$$\begin{cases} P(X,n)=-4X^2+2nX-2n+8 \\ Q(X,n)=-4X^2n+6n-n^2-32. \end{cases}$$

5. Application of the Boubaker-Turki Polynomials in Applied Physics Problems

5.1. Magnetic resonance blood flow imaging

It is known, as confirmed by M.B. Scheidegger et al. (1992), P. Boesiger et al. (1992), Liu (1992), Stahlberg (1992) and Schmalbrock (1990), that there are three main ways of generating magnetic resonance (MR) signals: Free induction decay (FID), Spin Echo (SE), and Gradient Echo technique. Integrating these signal types with the gradient sequences necessary for spatial encoding produces the fundamental magnetic resonance imaging sequences. All other imaging Magnetic resonance sequences can be characterized using these three basic ways of MR signal generation. In the following discussion, attention will be focused on quantitative blood flow imaging based on the phase contrast method.

Phase contrast technique employs the phase shift in the MR signal that is induced by the flowing blood. Blood spins moving along an applied gradient acquire a phase shift which is proportional to the strength and duration of the gradient and the motion of the spins. Therefore, with phase contrast method, complete suppression of stationary tissue can be achieved. This means that small blood vessels can be clearly visualized, even with slowly flowing blood.

5.2. Spatial magnetization equation

The Spatial magnetization component M is a solution of the equation (15) as mentioned by Awojoyogbe (2002, 2003, 2004):

$$v^2\frac{\partial^2 M}{\partial x^2}+\left(\frac{1}{T_1}+\frac{1}{T_2}\right)v\frac{\partial M}{\partial x}+\left(\gamma^2 B_1^2(x,t)+\frac{1}{T_1T_2}\right)M=\frac{M_o\gamma B_1(x.t)}{T_1}, \tag{15}$$

where γ is the gyromagnetic ratio of fluid spins, $B_1(x,t)$ is the rotating field, T_1 & T_2 are the spin lattice and spin-spin relaxation times respectively and v is the fluid velocity.

If we assume that the rotating field, which is a controlled item, can be expressed as a proportional function to the right term in the expression (14), the m-Boubaker polynomials in equation (14) are consequently solutions of a general second order non homogeneous differential equation (15) derived from the Bloch NMR flow equation.

Identification of the left term of equation (15) allows assuming that we can write (16)

$$\begin{cases} \tilde{B}_o \equiv \left(\dfrac{M_o\gamma}{T_1}\right)^{-1}\left(\gamma^2 B_1^{\,2}(x,t)+\dfrac{1}{T_1T_2}\right), & \text{position of spins} \\ \tilde{B}_1 \equiv \left(\dfrac{M_o\gamma}{T_1}\right)^{-1}\left(\dfrac{1}{T_1}+\dfrac{1}{T_2}\right)v, & \text{velocity of blood flow} \\ \tilde{B}_2 \equiv \left(\dfrac{M_o\gamma}{T_1}\right)^{-1} v^2, & \text{acceleration of blood flow} \\ \tilde{B}_3, \tilde{B}_4, \tilde{B}_5, \tilde{B}_6, \tilde{B}_7 \cdots. & \text{the higher order terms of the phase of the MRI signal.} \end{cases} \tag{16}$$

Hence, the properties of the established generating function (9), the Boubaker-Turki polynomials and the Bloch NMR flow equations discussed above can be very significant to extract relevant flow parameters for quantitative analysis of blood flow especially under pathological conditions

5.3. A Solution to the Heat Transfer Equation

Ghanouchi et al. (2007) proposed a solution of a heat transfer problem in the case of a modulated heat supply targeting a plate surface. In this study, the source term in the main heat transfer equation was decomposed in a spectral domain of Boubaker-Turki polynomials. The authors took benefit from the arithmetic and differential properties of these polynomials in order to discuss a particular problem that several precedent studies tried to explain: the establishment of non-Gaussian isothermal generative lines beneath a plate surface targeted by a Gaussian beam.

6. Conclusion

The present work is a continuation of the previous publications on the Boubaker polynomials. The obtained differential equation was inhomogeneous with a Chebyshev-dependent second term. This equation can be a first order supply to investigations of mathematical models involving mathematical modeling of physical and biophysical systems like the studies published by Riahi (2007), Slama et al. (2008), Srivastava (2007) and Marik (2006). Further investigations are focused on properties that lead to a characteristic homogenous equation. It was interesting to note that equations (9), (14) and (15) can play fundamental roles in the search for the best possible data obtainable on highly complex blood flow conditions. This will be the focus of our next investigation. It may be particularly interesting to note that the NMR flow parameters such as the flow velocity, T_1 & T_2 NMR relaxation parameters for a sample can be uniquely determined through the first few m-Boubaker polynomials.

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