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Solving Higher Dimensional Initial Boundary Value Problems by Variational Iteration Decomposition Method

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Abstract

In this paper, we apply a relatively new technique which is called the variational iteration decomposition method (VIDM) by combining the traditional variational iteration and the decomposition methods for solving higher dimensional initial boundary value problems. The proposed method is an elegant combination of variational iteration and the decomposition methods. The analytical results of the problems have been obtained in terms of convergent series with easily computable components. The method is quite efficient and is practically well suited for use in these problems. Several examples are given to verify the accuracy and efficiency of the proposed technique.

Keywords: Variational iteration method, nonlinear problems, initial value problems, boundary value problems, Adomian's polynomials

AMS 2000 Subject Classification Numbers: 65 N 10, 34 Bxx

1. Introduction

The numerical and analytical solutions of higher dimensional initial boundary value problems of variable coefficients, linear and nonlinear, are of considerable significance for applied sciences. Examples of linear models are Euler-Darboux equation, Lambropoubs' equation, and Tricomi equation, (see Miller (1973), Soliman (2006), Noor and Mohyud-Din (2008f), Wazwaz (1999), Wilcox (1970)) given by:

$$(x - y)u_{xy} + (\alpha u_x - \beta u_y) = 0,$$

$$u_{xy} + axu_x + byu_y + cxyu + u_t = 0,$$

$$u_{yy} = yu_{xx},$$

respectively. Examples of nonlinear models are introduced in KdV equation of variable coefficients and Clairaut's equation, (see) given by

$$u_t + \alpha t^n u u_x + \beta t^m u_{xxx} = 0,$$

$$u = xu_x + yu_y + f(u_x, u_y),$$

respectively. In this paper, we consider the higher dimensional initial boundary value problem of the form

$$u_{tt} = F(x, y, z, u, u_{xx}^m, u_{yy}^m, u_{zz}^m), \quad 0 < x < z, \quad 0 < y < b, \quad 0 < z < c, \quad m \geq 2, \quad t > 0,$$

subject to the Dirichlet boundary conditions

$$\begin{aligned} u(0, y, z, t) &= f_1(y, z, t), & u(a, y, z, t) &= f_2(y, z, t), & u(x, 0, z, t) &= g_1(x, z, t), \\ u(x, b, z, t) &= g_2(x, z, t), & u(x, y, 0, t) &= h_1(x, y, t), & u(x, y, c, t) &= h_2(x, y, t), \end{aligned}$$

and the initial conditions

$$u(x, y, z, 0) = \phi(x, y, z), \quad u_t(x, y, z, 0) = \eta(x, y, z),$$

where F is a linear or a nonlinear function. A considerable size of research work has been invested in these scientific applications. Several techniques including the spectral, characteristics, decomposition and variational homotopy perturbation have been used for solving these problems, (see Miller (1973), Soliman (2006), Noor and Mohyud-Din (2008f), Wazwaz (1999), Wilcox (1970)). He developed the variational iteration method for solving linear, nonlinear, initial and boundary value problems, (see He ((1999), He (2000), He (2006), He (2007a)).

It is worth mentioning that the origin of variational iteration method can be traced back to Inokuti, Sekine and Mura, (see Inokuti, Sekine and Mura (1978)) but the true potential of the VIM was explored by He. Since the beginning of 1980s, the Adomian's decomposition method has been applied to a wide class of functional equations, (see Wazwaz (1999)). In these methods the solution is given in an infinite series usually converging to an accurate solution.

Inspired and motivated by the ongoing research in this area, we apply the variational iteration decomposition method (VIDM) for solving the higher dimensional initial boundary value problems. This method is an elegant combination of variational iteration and the Adomian's decomposition methods for solving higher dimensional initial boundary value problems. This idea has been used implicitly by Abbasbandy, (see Abbasbandy (2007a), Abbasbandy (2007b)) for solving quadratic Riccati differential and Klein-Gordon equations.

In a subsequent work, Noor and Mohyud-Din developed the elegant coupling of Adomian's polynomials with the correctional functional of variational iteration method for solving various classes of singular and nonsingular initial and boundary value problems, (see Mohyud-Din, Noor & Noor (2008a), Noor and Mohyud-Din (2007c), Noor and Mohyud-Din (2008h)). Several examples are given to illustrate the reliability and performance of the proposed method.

2. Variational Iteration Method

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \quad (1)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method, (see He ((1999), He (2000), He (2006), He (2007a), He (2007b), Abbasbandy (2007a), Abbasbandy (2007b), Mohyud-Din, Noor& Noor (2008a), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b), Noor and Mohyud-Din (2008b), Noor and Mohyud-Din (2008c), Noor and Mohyud-Din (2008c), Noor and Mohyud-Din (2008c), Noor and Mohyud-Din (2008d), Noor and Mohyud-Din (2008e), Noor and Mohyud-Din (2008f), Noor and Mohyud-Din (2008g)), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s)) ds, \quad (2)$$

Where λ is a Lagrange multiplier, (see), which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, \tilde{u}_n is considered as a restricted variation, i.e., $\delta \tilde{u}_n = 0$. The relation (2) is called as a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given, (see He ((1999), He (2000), He (2006), He

(2007a), He (2007b)). In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1} , $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 , consequently, the solution is given by $u = \lim_{n \rightarrow \infty} u_n$.

3. Adomian's Decomposition Method

Consider the differential equation, (see Wazwaz (1999))

$$Lu + Ru + Nu = g, \quad (3)$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L , Nu represents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to both sides of (3) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu),$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method, (see Wazwaz (1999)) defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$\begin{aligned} u_0 &= f, \\ u_{k+1} &= L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0. \end{aligned}$$

The nonlinear operator $N(u)$ can be decomposed into an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in, (see Wazwaz (1999)) which yields

$$A_n = \left(\frac{1}{n!} \right) \left(\frac{d^n}{d\lambda^n} \right) N \left(\sum_{i=0}^n (\lambda^i u_i) \right)_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

4. Variational Iteration Decomposition Method (VIDM)

To illustrate the variational iteration decomposition method, we introduce Adomian's polynomials in the correction functional (2). Hence, we obtain the following iterative scheme for finding the approximate solutions

$$u^{(n+1)}(x) = u^{(n)}(x) + \int_0^t \lambda (L u^{(n)}(x) + \sum_{n=0}^{\infty} A_n - g(x)) dx.$$

This method is called as the variational iteration decomposition method (VIDM) and is obtained by the elegant coupling of Adomian's polynomials and the correctional functional of variational iteration method, (see Mohyud-Din, Noor & Noor (2008a), Noor and Mohyud-Din (2008f), Noor and Mohyud-Din (2008h), Noor and Mohyud-Din (2007c)).

5. Numerical Applications

In this section, we apply the variational iteration decomposition method (VIDM) for solving the higher dimensional initial boundary value problems. For the sake of comparison we consider the same examples as discussed in Wazwaz (999) and Noor and Mohyud-Din (2008f).

Example 5.1.

Consider the two dimensional initial boundary value problem

$$u_{tt} = \frac{1}{2} y^2 u_{xx} + \frac{1}{2} x^2 u_{yy}, \quad 0 < x, y < 1, t > 0,$$

with boundary conditions

$$\begin{aligned} u(0, y, t) &= y^2 e^{-t}, & u(1, y, t) &= (1 + y^2) e^{-t}, \\ u(x, 0, t) &= y^2 e^{-t}, & u(x, 1, t) &= (1 + x^2) e^{-t}, \end{aligned}$$

and the initial conditions

$$u(x, y, 0) = x^2 + y^2, \quad u_t(x, y, 0) = -(x^2 + y^2).$$

The correction functional is given as

$$u_{n+1}(x, y, z, t) = (x^2 + y^2) - (x^2 + y^2)t + \int_0^t \lambda(\xi) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{2} (y^2 (\tilde{u}_n)_{xx} + x^2 (\tilde{u}_n)_{yy}) \right) d\xi.$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - t$, yields the following iteration formula

$$u_{n+1}(x, y, z, t) = (x^2 + y^2) - (x^2 + y^2)t + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{2} (y^2 (u_n)_{xx} + x^2 (u_n)_{yy}) \right) d\xi.$$

Consequently, the following approximants are obtained

$$\begin{aligned} u_0(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t, \\ u_1(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2) \frac{t^2}{2!} - (x^2 + y^2) \frac{t^3}{3!}, \\ u_2(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2) \frac{t^2}{2!} - (x^2 + y^2) \frac{t^3}{3!} + (x^2 + y^2) \frac{t^4}{4!} - (x^2 + y^2) \frac{t^5}{5!}, \\ u_3(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2) \frac{t^2}{2!} - (x^2 + y^2) \frac{t^3}{3!} + (x^2 + y^2) \frac{t^4}{4!} - (x^2 + y^2) \frac{t^5}{5!} \\ &\quad + (x^2 + y^2) \frac{t^6}{6!} - (x^2 + y^2) \frac{t^7}{7!}, \\ &\vdots \end{aligned}$$

The series solution is given by

$$u(x, y, t) = (x^2 + y^2) \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \frac{t^8}{8!} - \dots \right),$$

and in a closed form by:

$$u(x, y, t) = (x^2 + y^2) e^{-t}.$$

Example 5.2.

Consider the three dimensional initial boundary value problem

$$u_{tt} = \frac{1}{45}x^2u_{xx} + \frac{1}{45}y^2u_{yy} + \frac{1}{45}z^2u_{zz} - u, \quad 0 < x, y < 1, t < 0,$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0, y, z, t) &= 0, & u_x(1, y, z, t) &= 6y^6z^6 \sinh t, & u_y(x, 0, z, t) &= 0, \\ u_y(x, 1, z, t) &= 6x^6z^6 \sinh t, & u_z(x, y, 0, t) &= 0, & u_z(x, y, 1, t) &= 6x^6y^6 \sinh t, \end{aligned}$$

and the initial conditions

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^6y^6z^6.$$

The correction functional is given as

$$u_{n+1}(x, y, z, t) = (x^6y^6z^6)t + \int_0^t \lambda(\xi) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{45} (x^2 (\tilde{u}_n)_{xx} + y^2 (\tilde{u}_n)_{yy} + z^2 (\tilde{u}_n)_{zz}) + \tilde{u}_n \right) d\xi.$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - t$, yields the following iteration formula

$$u_{n+1}(x, y, z, t) = (x^6y^6z^6)t + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} - \frac{1}{45} (x^2 (u_n)_{xx} + y^2 (u_n)_{yy} + z^2 (u_n)_{zz}) + u_n \right) d\xi.$$

Consequently, the following approximants are obtained

$$\begin{aligned} u_0(x, y, z, t) &= x^6y^6z^6 t, \\ u_1(x, y, z, t) &= x^6y^6z^6 \left(t + \frac{t^3}{3!} \right), \\ u_2(x, y, z, t) &= x^6y^6z^6 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} \right), \\ u_3(x, y, z, t) &= x^6y^6z^6 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} \right), \\ u_4(x, y, z, t) &= x^6y^6z^6 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} \right), \\ &\vdots \end{aligned}$$

The series solution is given by

$$u(x, y, z, t) = x^6 y^6 z^6 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \frac{t^9}{9!} + \dots \right),$$

and in a closed form by

$$u(x, y, z, t) = x^6 y^6 z^6 \sinh t.$$

Example 5.3.

Consider the two dimensional nonlinear inhomogeneous initial boundary value problem

$$u_{tt} = 2x^2 + 2y^2 + \frac{15}{2}(xu_{xx}^2 + yu_{yy}^2), \quad 0 < x, y < 1, t > 0$$

with boundary conditions

$$\begin{aligned} u(0, y, t) &= y^2 t^2 + y t^6, & u(1, y, t) &= (1 + y^2) t^2 + (1 + y) t^6, \\ u(x, 0, t) &= x^2 t^2 + x t^6, & u(x, 1, t) &= (1 + x^2) t^2 + (1 + x) t^6, \end{aligned}$$

and the initial conditions

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0.$$

The correction functional is given as

$$u_{n+1}(x, y, t) = \int_0^t \lambda(\xi) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{15}{2} \left(x(\tilde{u}_n)_{xx}^2 + y(\tilde{u}_n)_{yy}^2 \right) + (2x^2 + 2y^2) \right) d\xi,$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - t$, yields the following iteration formula

$$u_{n+1}(x, y, t) = \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{15}{2} \left(x(u_n)_{xx}^2 + y(u_n)_{yy}^2 \right) + (2x^2 + 2y^2) \right) d\xi.$$

Applying the variational iteration decomposition method

$$u_{n+1}(x, y, t) = \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \frac{15}{2} \left(x \sum_{n=0}^{\infty} Q_n + y \sum_{n=0}^{\infty} M_n \right) + (2x^2 + 2y^2) \right) d\xi,$$

where Q_n and M_n are the so-called Adomian's polynomials that represent the nonlinear operators u_{xx}^2 , u_{yy}^2 and can be generated for all type of nonlinearities, (see Wazwaz (1999)) which yields the following

$$\begin{aligned} Q_0 &= u_{0,xx}^2, & Q_1 &= 2u_{0,xx}^2 u_{1,xx}, & Q_2 &= 2u_{0,xx}^2 u_{2,xx} + u_{1,xx}^2, & Q_3 &= 2u_{0,xx}^2 u_{3,xx} + 2u_{1,xx} u_{2,xx}, \\ &\vdots & & & & & & \\ M_0 &= u_{0,yy}^2, & M_1 &= 2u_{0,yy}^2 u_{1,yy}, & M_2 &= 2u_{0,yy}^2 u_{2,yy} + u_{1,yy}^2, & M_3 &= 2u_{0,yy}^2 u_{3,yy} + 2u_{1,yy} u_{2,yy}, \\ &\vdots & & & & & & \end{aligned}$$

Consequently, the following approximants are obtained

$$\begin{aligned} u_0(x, y, t) &= 0, \\ u_1(x, y, t) &= (x^2 + y^2)t^2, \\ u_2(x, y, t) &= (x^2 + y^2)t^2 + (x + y)t^6, \\ u_3(x, y, t) &= 0, \\ &\vdots \end{aligned}$$

The exact solution is obtained as

$$u(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6.$$

Example 5.4.

Consider the three dimensional nonlinear initial boundary value problem

$$u_{tt} = (2 - t^2) + u - (e^{-x}u_{xx}^2 + e^{-y}u_{yy}^2 + e^{-z}u_{zz}^2), \quad 0 < x, y < 1, t < 0$$

subject to the Neumann boundary conditions

$$\begin{aligned} u_x(0, y, z, t) &= 1, & u_x(1, y, z, t) &= e, & u_y(x, 0, z, t) &= 0, \\ u_y(x, 1, z, t) &= 6x^6 z^6 \sinh t, & u_z(x, y, 0, t) &= 1, & u_z(x, y, 1, t) &= e, \end{aligned}$$

and the initial conditions

$$u(x, y, z, 0) = e^x + e^y + e^z, \quad u_t(x, y, z, 0) = 0.$$

The correction functional is given as

$$u_{n+1}(x, y, z, t) = (e^x + e^y + e^z) + \int_0^t \lambda(\xi) \left(\frac{\partial^2 u_n}{\partial \xi^2} + (e^{-x} (\tilde{u}_n)_{xx}^2 + e^{-y} (\tilde{u}_n)_{yy}^2 + e^{-z} (\tilde{u}_n)_{zz}^2) + \tilde{u}_n \right) d\xi \\ + \int_0^t \lambda(\xi) (2 - t^2) d\xi.$$

Making the above functional stationary, the Lagrange multiplier can be determined as $\lambda = \xi - t$, yields the following iteration formula

$$u_{n+1}(x, y, z, t) = (e^x + e^y + e^z) + \int_0^t (\xi - t) \left(\frac{\partial^2 u_n}{\partial \xi^2} + \left(e^{-x} \sum_{n=0}^{\infty} M_n + e^{-y} \sum_{n=0}^{\infty} Q_n + e^{-z} \sum_{n=0}^{\infty} H_n \right) \right) d\xi \\ + \int_0^t (\xi - t) \left(\sum_{n=0}^{\infty} u_n + (2 - t^2) \right) d\xi.$$

where Q_n , M_n and H_n are the so-called Adomian's polynomials that represent the nonlinear operators u_{xx}^2 , u_{yy}^2 , u_{zz}^2 and can be generated for all type of nonlinearities, (see Wazwaz (1999)). Consequently, the following approximants are obtained

$$u_0(x, y, z, t) = (e^x + e^y + e^z) + t^2 - \frac{t^4}{12}, \\ u_1(x, y, z, t) = (e^x + e^y + e^z) + t^2 - \frac{t^4}{12} + \frac{t^4}{12} - \frac{t^6}{360}, \\ u_2(x, y, z, t) = (e^x + e^y + e^z) + t^2 - \frac{t^4}{12} + \frac{t^4}{12} - \frac{t^6}{360} + \frac{t^6}{360} - \frac{t^8}{20160}, \\ \vdots$$

Using the noise terms phenomena, we obtained

$$u(x, y, z, t) = (e^x + e^y + e^z) + t^2.$$

6. Conclusions

In this paper, we applied the variational iteration decomposition method (VIDM) for solving linear and nonlinear higher dimensional initial boundary value problems with constant coefficients, variable coefficients and the function coefficients. The proposed method is applied in a direct way without using linearization, perturbation or restrictive assumptions. In all the numerical results exact solutions were obtained by using the initial conditions only.

It may be concluded that method is very powerful and efficient in finding the analytical solutions for a wide class of initial boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus we conclude that the VIDM can be considered as an efficient alternative for solving initial and boundary value problems.

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