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Soliton Perturbation Theory for the Modified Kawahara Equation

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Abstract

The modified Kawahara equation is studied along with its perturbation terms. The adiabatic dynamics of the soliton amplitude and the velocity of the soliton are obtained by the aid of soliton perturbation theory.

Keywords: Kawahara equation, perturbation, soliton

AMS Codes: 35Q51, 35Q53, 37K10

1. Introduction

The theory of nonlinear evolution equations is an ongoing topic of research for decades 1to10. This paper is going to study one of the classical nonlinear evolution equations that is known as the modified Kawahara equation (mKE). The dimensionless form of the mKE that is going to be studied in this paper is given by

$$q_t + aq^2 q_x + bq_{xxx} - cq_{xxxxx} = 0 , \qquad (1)$$

where a , b and c are arbitrary constants. This dispersive equation was proposed by Kawahara in 1972 as an important dispersive equation that arises in the context of shallow water waves (Kawahara (1972)). The mKE given by (1) is not integrable by the classical method of Inverse Scattering Transform as this equation will fail the Painleve test of integrability. However, in the last few years, very powerful methods of integration of nonlinear evolution equations of this type were developed. They include the Wadati trace method, pseudo-spectral method, tanh-sech method, sine-cosine method and the Riccati equation expansion method (Chen (2007), Malfliet (1992), Parkes (1996), Wazwaz (2007)). It is to be noted that one of the major disadvantage of these modern methods of integrability is that one can only obtain the 1-soliton solution of such a nonlinear evolution equation and not a multi-soliton solution. Also these methods are unable to compute a closed form solution for the soliton radiation. Using the sine-cosine method, the 1-soliton solution of (1) is given by (Sirendaoreji (2004), Wazwaz (2007))

$$q(x,t) = \frac{A}{\cosh^2 B(x-\bar{x})}, \tag{2}$$

where,

$$A = -\frac{3b}{\sqrt{10ac}}, \tag{3}$$

$$B = \frac{\sqrt{b}}{2\sqrt{5c}}. \tag{4}$$

Here A represents the amplitude of the soliton, while B is the inverse width of the soliton and $\bar{x}$ represents the center position of the soliton and therefore the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt}. \tag{5}$$

2. Mathematical Properties

Equation (1) has at least two integrals of motion (Zhidkov (2001)) that are known as linear momentum (M) and energy (E). These are respectively given by:

$$M = \int_{-\infty}^{\infty} q\,dx = 4A = -\frac{12b}{\sqrt{10ac}} \tag{6}$$

and

$$E = \int_{-\infty}^{\infty} q^2\,dx = \frac{8}{3}A^2 = \frac{12b^2}{5ac}. \tag{7}$$

These conserved quantities are calculated by using the 1-soliton solution given by (2). The center of the soliton $\bar{x}$ is given by the definition

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xqdx}{\int_{-\infty}^{\infty} qdx} = \frac{\int_{-\infty}^{\infty} xqdx}{M}, \tag{8}$$

where M is defined in (6). Thus, the velocity of the soliton is given by

$$v = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} xq_t dx}{\int_{-\infty}^{\infty} qdx} = \frac{\int_{-\infty}^{\infty} xq_t dx}{M}. \tag{9}$$

On using (1) and (9), the velocity of the soliton reduces to

$$v = \frac{4b^2}{25c}. \tag{10}$$

3. Perturbation Terms

The perturbed mKE that is going to be studied in this paper is given by

$$q_t + aq^2 q_x + bq_{xxx} - cq_{xxxxx} = \varepsilon R, \tag{11}$$

where in (11), ε is the perturbation parameter and $0 < \varepsilon << 1$ (Biswas (2006), Kivshar (1989), Osborne (1997)), while R gives the perturbation terms. In presence of perturbation terms, the momentum and the energy of the soliton do not stay conserved. Instead, they undergo adiabatic changes that lead to the adiabatic deformation of the soliton amplitude, width and a slow change in the velocity (Kivshar (1989), Osborne (1997)). Using (7), the law of adiabatic deformation of the soliton energy is given by (Biswas (2006), Chen (2007), Kawahara (1972), Kivshar (1989))

$$\frac{dE}{dt} = 2\varepsilon \int_{-\infty}^{\infty} xRdx, \tag{12}$$

while the adiabatic law of change of the velocity of the soliton is given by (Biswas (2006), Chen (2007), Kawahara (1972), Kivshar (1989))

$$v = \frac{4b^2}{25c} + \frac{\varepsilon}{M} \int_{-\infty}^{\infty} xRdx. \tag{13}$$

In order to obtain (12), equation (11) is first multiplied both sides by q and then integrated with respect to x. Since for solitons, q, q_x, q_{xx}, q_{xxx} etc. all approach zero as x approaches $\pm\infty$, it is only the first term in (11) that sustains, that leads to (12). Also, in order to obtain (13), equation

(9) is utilized where qt is replaced by all the terms in the right hand side of equation (11) and the same technique is applied that leads to (13).

3.1.Examples

In this paper, the perturbation terms that are going to be considered are

$$R = \alpha q + \beta q_{xx} + \chi q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \upsilon q q_x q_{xx} + \sigma q_x^3 + \varepsilon q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}. \tag{14}$$

So, the perturbed mKE that is going to be considered in this paper is

$$q_t + aq^2 q_x + bq_{xxx} - cq_{xxxxx} = \varepsilon [\alpha q + \beta q_{xx} + \chi q_x q_{xx} + \delta q^m q_x + \lambda q q_{xxx} + \upsilon q q_x q_{xx} + \sigma q_x^3 + \varepsilon q_x q_{xxxx} + \eta q_{xx} q_{xxx} + \rho q_{xxxx} + \psi q_{xxxxx} + \kappa q q_{xxxxx}]. \tag{15}$$

The perturbation terms due to α appear due to shoaling and β is a dissipative term (Chen (2007)). The coefficient of δ is the higher nonlinear dispersion while the coefficient of Ψ represents the higher spatial dispersion. In (14), m is a positive integer and $1 \leq m \leq 4$. The term with the coefficient of ρ will provide the higher stabilizing term and must therefore be taken into account while Ψ is the coefficient of higher order dispersion. The remaining coefficients appear in the context of Whitham hierarchy (Parkes (1996)).

3.2. Applications

In presence of these perturbation terms, the adiabatic variation of the energy of the soliton is given by:

$$\frac{dE}{dt} = \frac{16\varepsilon A^2}{105}(35\alpha - 7\beta + 5\rho). \tag{16}$$

Using (7), one can integrate equation (16) to yield

$$A = A_0 e^{\frac{\varepsilon}{35}(35\alpha - 7\beta + 5\rho)t}, \tag{17}$$

where A_0 is the initial amplitude of the soliton. This leads to the long term behavior of the soliton amplitude as

$$\lim_{t\to\infty} A(t) = \begin{cases} A_0, & : 7\beta = 35\alpha + 5\rho \\ \infty, & : 7\beta < 35\alpha + 5\rho \\ 0, & : 7\beta > 35\alpha + 5\rho. \end{cases} \tag{18}$$

The law of the change of velocity for the given perturbation terms in (14) is

$$v = \frac{4b^2}{25c} - \varepsilon\left[\frac{m\delta A^m}{(m+1)(2m+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(m)}{\Gamma\left(m+\frac{1}{2}\right)}\right]\frac{A}{315}\left\{3(7\gamma - 14\lambda - 15\xi + 5\eta + 25\kappa) + 2vA\right\}. \quad (19)$$

In order to evaluate (16) and (19), the 1-soliton solution given by (2) is substituted in (12) and (13) respectively. Although, technically, it is improper to substitute the unperturbed 1-soliton solution given by (2) into (12) and (13), this is only an approximate result and this is the technique that is widely used in the literature of soliton theory (Biswas (2006)). It is the radiation term that is not taken into consideration that makes this technique approximate.

It is to be noted that in the evaluation of the adiabatic variation of the energy in (16), the integrals vanish for all the perturbation terms except α , β and ρ . The remaining terms vanish because of the fact that those terms lead to an integrand that is an odd function. A similar situation is valid in the evaluation of the soliton velocity change in (19).

4. Conclusions

In this paper, soliton perturbation theory is used to study the perturbed mKE. This theory gives the ability to compute the adiabatic variation of the soliton energy and hence the adiabatic variation of the soliton amplitude. This finally leads to the computation of the long term behavior of the soliton amplitude depending on the specific combination of the soliton parameters. Also, it is shown that the velocity undergoes a slow change due to these perturbation terms.

In future, the integration of the perturbed mKE will be carried out by the aid of multiple-scale perturbation analysis. Thus, the quasi-stationary soliton (Biswas (2006)), in presence of such perturbation terms, will be obtained. These results will be reported in a future publication.

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