



On a -ary Subdivision for Curve Design II. 3-Point and 5-Point Interpolatory Schemes

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Abstract

The a -ary 3-point and 5-point interpolatory subdivision schemes for curve design are introduced for arbitrary odd integer $a \geq 3$. These new schemes further extend the family of the classical 4- and 6-point interpolatory schemes.

Keywords: Subdivision; Curve design; Stationary; Refinable functions; a -ary

MSC 2000: 14H50; 17A42; 65D17; 68U07

1. Introduction

THIS is a continuation of (Lian [4]), where the classical 4- and 6-point binary interpolatory subdivision schemes for curve design in (Dyn, et al. [1]) and (Weissman [5]) were extended to a -ary interpolatory schemes for any $a \geq 3$.

One of the main objectives of the current paper is to introduce and extend both the 4- and 6-point a -ary interpolatory schemes further to the 3- and 5-point a -ary interpolatory schemes for any odd $a \in \mathbb{Z}_+$ with $a \geq 3$. Similar to the 4- and 6-point a -ary schemes, we also require the refinable functions corresponding to the 3- and 5-point a -ary interpolatory schemes have polynomial preservation orders of 3 and 5, respectively, or ${}^a\phi_3 \in \mathbb{PP}_3$ and ${}^a\phi_5 \in \mathbb{PP}_5$ for short. Observe that, when $a \geq 2$ is even, for either ${}^a\phi_3 \in \mathbb{PP}_3$ or ${}^a\phi_5 \in \mathbb{PP}_5$, the interpolatory property and the symmetry on either ${}^a\phi_3$ or ${}^a\phi_5$ are not compatible. That is exactly the reason why the

dilation factor a has to be odd now.

Our main results are, listed in Section 2, the explicit expressions of two-scale symbols of both ${}^a\phi_3 \in \mathbb{PP}_3$ and ${}^a\phi_5 \in \mathbb{PP}_5$. Their proofs are given in Section 3. Some applications to curve design are demonstrated in Section 4. A few remarks and future work constitute Section 5.

2. Main Results

Let ${}^a\phi_3$ and ${}^a\phi_5$ be the scaling functions with odd dilation factor $a \geq 3$, which correspond the 3- and 5-point interpolatory subdivision schemes for curve design. For ${}^a\phi_3$, we have the following.

Theorem 1: The scaling function ${}^a\phi_3 \in \mathbb{PP}_3$ with the smallest support, is determined from the two-scale symbol aP_3 of the form

$${}^aP_3(z) = z^{(1-3a)/2} \left(\frac{1}{a} \frac{1-z^a}{1-z} \right)^3 \left(\frac{1-a^2}{8} + \frac{3+a^2}{4}z + \frac{1-a^2}{8}z^2 \right). \quad (1)$$

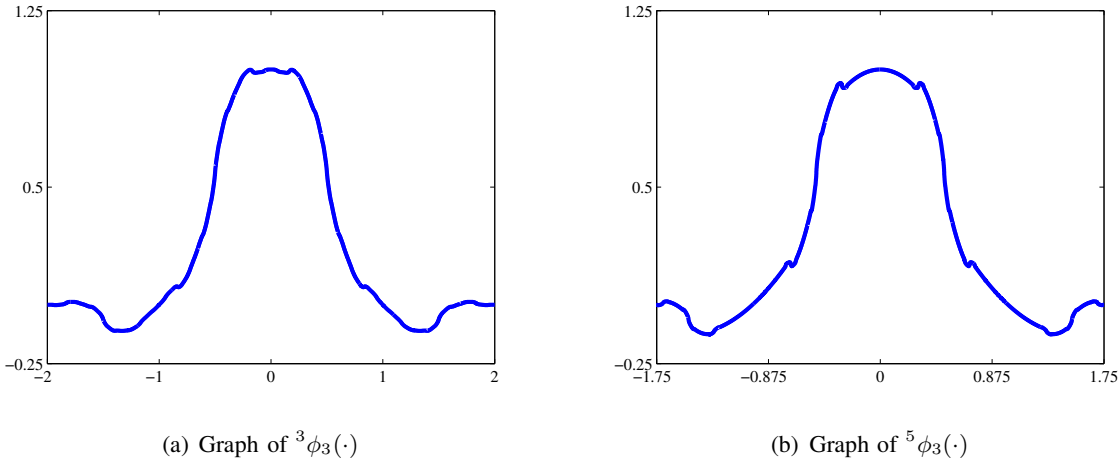


Fig. 1. The interpolatory scaling functions ${}^3\phi_3(\cdot)$ and ${}^5\phi_3(\cdot)$ determined from the two-scale equations in (1) when $a = 3$ and 5, where $\text{supp } {}^3\phi_3 = [-2, 2]$ and $\text{supp } {}^5\phi_3 = [-7/4, 7/4]$, respectively.

See Fig. 1 for the graphs of ${}^3\phi_3$ and ${}^5\phi_3$. For ${}^a\phi_5 \in \mathbb{PP}_5$, we have the following.

Theorem 2: The scaling function ${}^a\phi_5 \in \mathbb{PP}_5$ with the smallest support, is determined from the two-scale symbol aP_5 of the form

$${}^aP_5(z) = z^{(1-5a)/2} \left(\frac{1}{a} \frac{1-z^a}{1-z} \right)^5 \left[\frac{(a-1)(a+1)(3a-1)(3a+1)}{384} - \frac{(a-1)(a+1)(9a^2+19)}{96}z + \frac{115+50a^2+27a^4}{192}z^2 - \frac{(a-1)(a+1)(9a^2+19)}{96}z^3 + \frac{(a-1)(a+1)(3a-1)(3a+1)}{384}z^4 \right]. \quad (2)$$

See Fig. 2 for the graphs of ${}^3\phi_5$ and ${}^5\phi_5$. It is also easy to verify that

$$\text{supp } {}^a\phi_3 = \left[-\frac{3a-1}{2(a-1)}, \frac{3a-1}{2(a-1)} \right], \quad \text{supp } {}^a\phi_5 = \left[-\frac{5a-1}{2(a-1)}, \frac{5a-1}{2(a-1)} \right].$$

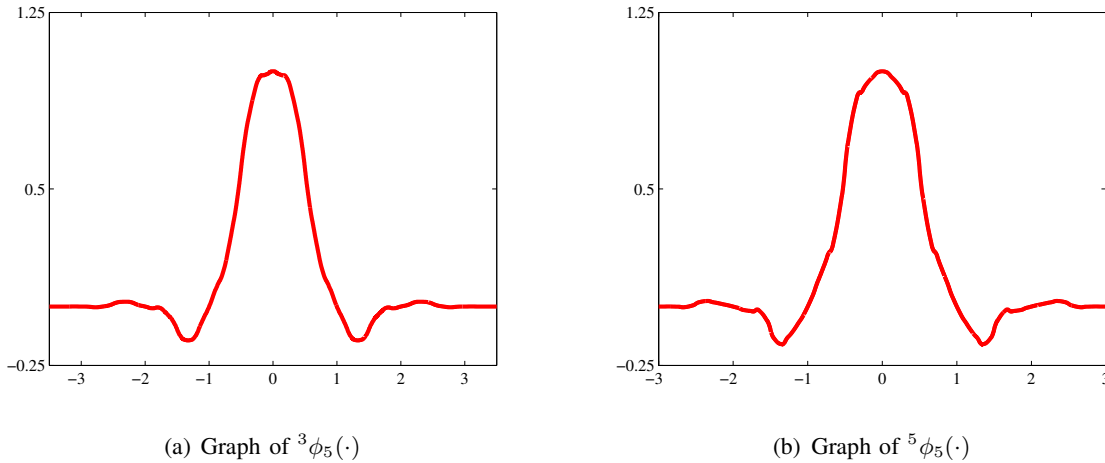


Fig. 2. The interpolatory scaling functions ${}^3\phi_5(\cdot)$ and ${}^5\phi_5(\cdot)$ determined from the two-scale equations in (2) when $a = 3$ and 5, where $\text{supp } {}^3\phi_5 = [-7/2, 7/2]$ and $\text{supp } {}^5\phi_5 = [-3, 3]$, respectively.

Indeed, if $\text{supp } {}^a\phi_3 = [\ell_3, r_3]$, it follows from (1) that the left-most contribution to ${}^a\phi_3(x)$ is ${}^a\phi_3\left(ax + \frac{3a-1}{2}\right)$ while the right-most contribution to ${}^a\phi_3(x)$ is ${}^a\phi_3\left(ax - \frac{3a-1}{2}\right)$. Hence, $\ell_3 \leq ax + \frac{3a-1}{2}$ and $ax - \frac{3a-1}{2} \leq r_3$, which leads to

$$\frac{1}{a} \left(\ell_3 - \frac{3a-1}{2} \right) = \ell_3, \quad \frac{1}{a} \left(r_3 + \frac{3a-1}{2} \right) = r_3,$$

so that $\ell_3 = -\frac{3a-1}{2(a-1)}$ and $r_3 = \frac{3a-1}{2(a-1)}$. Meanwhile, if $\text{supp } {}^a\phi_5 = [\ell_5, r_5]$, completely analogous process leads to $\ell_5 = -r_5 = -\frac{5a-1}{2(a-1)}$.

TABLE I
WEIGHTS OF a -ARY 3-POINT SUBDIVISION SCHEME

	$\lambda_{k-1}^{(n)}$	$\lambda_k^{(n)}$	$\lambda_{k+1}^{(n)}$
$\lambda_{ak-(a-1)/2}^{(n+1)}$	$\frac{a}{3}p^{(a+1)/2}$	$\frac{a}{3}p^{-(a-1)/2}$	$\frac{a}{3}p^{-(3a-1)/2}$
$\lambda_{ak-(a-3)/2}^{(n+1)}$	$\frac{a}{3}p^{(a+3)/2}$	$\frac{a}{3}p^{-(a-3)/2}$	$\frac{a}{3}p^{-(3a-3)/2}$
...
$\lambda_{ak-1}^{(n+1)}$	$\frac{a}{3}p^{a-1}$	$\frac{a}{3}p^{-1}$	$\frac{a}{3}p^{-a-1}$
$\lambda_{ak}^{(n+1)}$		1	
$\lambda_{ak+1}^{(n+1)}$	$\frac{a}{3}p^{a+1}$	$\frac{a}{3}p^1$	$\frac{a}{3}p^{-a+1}$
...
$\lambda_{ak+(a-3)/2}^{(n+1)}$	$\frac{a}{3}p^{(3a-3)/2}$	$\frac{a}{3}p^{(a-3)/2}$	$\frac{a}{3}p^{-(a+3)/2}$
$\lambda_{ak+(a-1)/2}^{(n+1)}$	$\frac{a}{3}p^{(3a-1)/2}$	$\frac{a}{3}p^{(a-1)/2}$	$\frac{a}{3}p^{-(a+1)/2}$

If we write aP_3 in (1) and aP_5 in (2) by

$${}^aP_3(z) = \frac{1}{a} \sum_{k=-3a+1}^{3a-1} {}^a_3p_k z^k, \quad {}^aP_5(z) = \frac{1}{a} \sum_{k=-5a+1}^{5a-1} {}^a_5p_k z^k,$$

the a -ary 3- and 5-point interpolatory subdivision schemes for curve design can be given by Table I and Table II, i.e., the 3-point scheme is given by

$$\lambda_{ak+\ell}^{(n+1)} = \sum_{j=-1}^1 {}^a_3p_{-aj+\ell} \lambda_{k+j}^{(n)}, \quad \ell = -(a-1)/2, \dots, (a-1)/2; \quad n \in \mathbb{Z}_+, \quad (3)$$

while the 5-point a -ary scheme is given by

$$\lambda_{ak+\ell}^{(n+1)} = \sum_{j=-2}^2 {}^a_5p_{-aj+\ell} \lambda_{k+j}^{(n)}, \quad \ell = -(a-1)/2, \dots, (a-1)/2, \quad n \in \mathbb{Z}_+. \quad (4)$$

The two-scale sequences $\{{}^a_3p_k\}_{k \in \mathbb{Z}}$ and $\{{}^a_5p_k\}_{k \in \mathbb{Z}}$ are listed explicitly in the following,

$${}^a_3p_{-k} = {}^a_3p_k = \frac{1}{a^2}(a+k)(a-k), \quad k = 0, \dots, (a-1)/2; \quad (5)$$

$${}^a_3p_{-k} = {}^a_3p_k = \frac{1}{2a^2}(a-k)(2a-k), \quad k = (a+1)/2, \dots, (3a-1)/2; \quad (6)$$

$${}^a_3p_k = 0, \quad |k| \geq (3a-1)/2, \quad (7)$$

and

$${}^a_5p_{-k} = {}^a_5p_k = \frac{1}{4a^4}(a-k)(a+k)(2a-k)(2a+k), \quad k = 0, \dots, (a-1)/2; \quad (8)$$

$${}^a_5p_{-k} = {}^a_5p_k = \frac{1}{6a^4}(a-k)(a+k)(2a-k)(3a-k), \quad k = (a+1)/2, \dots, (3a-1)/2; \quad (9)$$

$${}^a_5p_{-k} = {}^a_5p_k = \frac{1}{24a^4}(a-k)(2a-k)(3a-k)(4a-k), \quad k = (3a+1)/2, \dots, (5a-1)/2; \quad (10)$$

$${}^a_5p_k = 0, \quad |k| \geq (5a-1)/2. \quad (11)$$

The interpolatory property of both schemes in (3) and (4) follows from (5)–(7) and (8)–(11).

More explicitly, it follows from (3) and (5)–(7) that the 3-point a -ary interpolatory subdivision scheme is given by

$$\lambda_{ak-(a+1)/2+\ell}^{(n+1)} = \frac{(a+1-2\ell)(3a+1-2\ell)}{8a^2} \lambda_{k-1}^{(n)} + \frac{(a-1+2\ell)(3a+1-2\ell)}{4a^2} \lambda_k^{(n)} - \frac{(a+1-2\ell)(a-1+2\ell)}{8a^2} \lambda_{k+1}^{(n)}, \quad \ell = 1, \dots, (a-1)/2; \quad (12)$$

$$\lambda_{ak}^{(n+1)} = \lambda_k^{(n)}, \quad (13)$$

$$\lambda_{ak+\ell}^{(n+1)} = -\frac{\ell(a-\ell)}{2a^2} \lambda_{k-1}^{(n)} + \frac{(a-\ell)(a+\ell)}{a^2} \lambda_k^{(n)} + \frac{\ell(a+\ell)}{2a^2} \lambda_{k+1}^{(n)}, \quad \ell = 1, \dots, (a-1)/2. \quad (14)$$

TABLE II
WEIGHTS OF a -ARY 5-POINT SUBDIVISION SCHEME

	$\lambda_{k-2}^{(n)}$	$\lambda_{k-1}^{(n)}$	$\lambda_k^{(n)}$	$\lambda_{k+1}^{(n)}$	$\lambda_{k+2}^{(n)}$
$\lambda_{ak-(a-1)/2}^{(n+1)}$	$\frac{a}{5}P(3a+1)/2$	$\frac{a}{5}P(a+1)/2$	$\frac{a}{5}P-(a-1)/2$	$\frac{a}{5}P-(3a-1)/2$	$\frac{a}{5}P-(5a-1)/2$
$\lambda_{ak-(a-3)/2}^{(n+1)}$	$\frac{a}{5}P(3a+3)/2$	$\frac{a}{5}P(a+3)/2$	$\frac{a}{5}P-(a-3)/2$	$\frac{a}{5}P-(3a-3)/2$	$\frac{a}{5}P-(5a-3)/2$
...
$\lambda_{ak-1}^{(n+1)}$	$\frac{a}{5}P2a-1$	$\frac{a}{5}Pa-1$	$\frac{a}{5}P-1$	$\frac{a}{5}P-a-1$	$\frac{a}{5}P-2a-1$
$\lambda_{ak}^{(n+1)}$			1		
$\lambda_{ak+1}^{(n+1)}$	$\frac{a}{5}P2a+1$	$\frac{a}{5}Pa+1$	$\frac{a}{5}P1$	$\frac{a}{5}P-a+1$	$\frac{a}{5}P-2a+1$
...
$\lambda_{ak+(a-3)/2}^{(n+1)}$	$\frac{a}{5}P(5a-3)/2$	$\frac{a}{5}P(3a-3)/2$	$\frac{a}{5}P(a-3)/2$	$\frac{a}{5}P-(a+3)/2$	$\frac{a}{5}P-(3a+3)/2$
$\lambda_{ak+(a-1)/2}^{(n+1)}$	$\frac{a}{5}P(5a-1)/2$	$\frac{a}{5}P(3a-1)/2$	$\frac{a}{5}P(a-1)/2$	$\frac{a}{5}P-(a+1)/2$	$\frac{a}{5}P-(3a+1)/2$

Similarly, it is clear from (4) and (8)–(11) that the 5-point a -ary interpolatory subdivision scheme is explicitly given by

$$\begin{aligned}
 \lambda_{ak-(a+1)/2+\ell}^{(n+1)} = & -\frac{(a-1+2\ell)(a+1-2\ell)(3a+1-2\ell)(5a+1-2\ell)}{384a^4} \lambda_{k-2}^{(n)} \\
 & + \frac{(a+1-2\ell)(3a-1+2\ell)(3a+1-2\ell)(5a+1-2\ell)}{96a^4} \lambda_{k-1}^{(n)} \\
 & + \frac{(a-1+2\ell)(3a-1+2\ell)(3a+1-2\ell)(5a+1-2\ell)}{64a^4} \lambda_k^{(n)} \\
 & - \frac{(a+1-2\ell)(a-1+2\ell)(3a-1+2\ell)(5a+1-2\ell)}{96a^4} \lambda_{k+1}^{(n)} \\
 & + \frac{(a+1-2\ell)(a-1+2\ell)(3a+1-2\ell)(3a-1+2\ell)}{384a^4} \lambda_{k+2}^{(n)}, \\
 & \ell = 1, \dots, (a-1)/2; \tag{15}
 \end{aligned}$$

$$\lambda_{ak}^{(n+1)} = \lambda_k^{(n)}; \tag{16}$$

$$\begin{aligned}
 \lambda_{ak+\ell}^{(n+1)} = & \frac{\ell(a-\ell)(a+\ell)(2a-\ell)}{24a^4} \lambda_{k-2}^{(n)} - \frac{\ell(a-\ell)(2a-\ell)(2a+\ell)}{6a^4} \lambda_{k-1}^{(n)} \\
 & + \frac{(a-\ell)(a+\ell)(2a-\ell)(2a+\ell)}{4a^4} \lambda_k^{(n)} + \frac{\ell(a+\ell)(2a-\ell)(2a+\ell)}{6a^4} \lambda_{k+1}^{(n)} \\
 & - \frac{\ell(a-\ell)(a+\ell)(2a+\ell)}{24a^4} \lambda_{k+2}^{(n)}, \quad \ell = 1, \dots, (a-1)/2. \tag{17}
 \end{aligned}$$

We end this section by pointing out that the graphs of ${}^3\phi_3$ and ${}^5\phi_3$ in Fig. 1 and the graphs of ${}^3\phi_5$ and ${}^5\phi_5$ in Fig. 2 can also be obtained by the two subdivision schemes (12)–(14) and (15)–(17) with the initial sequence $\lambda_k^{(0)} = \delta_{k,0}, k \in \mathbb{Z}$.

3. Proofs of Main Results

Proof of Theorem 1.

First, an a -ary 3-point scheme needs at most $3a$ weights, i.e., the two-scale sequence $\{^a_3p_k\}_{k \in \mathbb{Z}}$ of $^a\phi_3$ has at most $3a$ consecutive nontrivial entries. Secondly, for $^a\phi_3$ to have the highest possible m of $\mathbb{P}\mathbb{P}_m$, its two-scale symbol aP_3 has to have the highest possible order of factor of $(1 + z + \cdots + z^{a-1})$. This leads to both $m = 3$ and aP_3 must have the form

$$^aP_3(z) = z^{(1-3a)/2} \left(\frac{1 - z^a}{1 - z} \right)^3 (s_0 + s_1z + s_2z^2)$$

for some constant s_0, s_1 , and s_2 satisfying $s_2 = s_0$ and $s_0 + s_1 + s_2 = 1$. By using $(1 - z)^{-3} = \sum_{\ell=0}^{\infty} \binom{2+\ell}{2} z^\ell$ we have

$$\begin{aligned} (s_0 + s_1z + s_2z^2) (1 - z)^{-3} &= \sum_{\ell=0}^{\infty} \mu_\ell z^\ell, \quad \text{where} \\ \mu_\ell &= \binom{\ell+2}{2} s_0 + \binom{\ell+1}{2} s_1 + \binom{\ell}{2} s_2, \quad \ell \in \mathbb{Z}_+. \end{aligned} \quad (18)$$

Hence, by defining $\mu_\ell = 0$ for all $\ell < 0$ and multiplying by the expansion of $(1 - z^a)^3$ we obtain the explicit expressions for $\{^a_3p_k\}_{k \in \mathbb{Z}}$ in terms of $\{\mu_\ell\}$, namely,

$$\begin{aligned} ^a_3p_k &= \frac{1}{a^2} (\mu_{(3a-1)/2+k} - 3\mu_{(a-1)/2+k} + 3\mu_{-(a+1)/2+k} - \mu_{-(3a+1)/2+k}), \\ k &= -(3a-1)/2, \dots, (3a-1)/2. \end{aligned} \quad (19)$$

Next, the three identities $^a_3p_{-a} = 0$, $^a_3p_0 = 1$, and $^a_3p_a = 0$, lead to

$$\begin{aligned} \frac{1}{a^2} \mu_{(a-1)/2} &= 0, \\ \frac{1}{a^2} (\mu_{(3a-1)/2} - 3\mu_{(a-1)/2}) &= 1, \\ \frac{1}{a^2} (\mu_{(5a-1)/2} - 3\mu_{(3a-1)/2} + 3\mu_{(a-1)/2}) &= 0, \end{aligned}$$

or simply $\mu_{(a-1)/2} = 0$, $\mu_{(3a-1)/2} = a^2$, $\mu_{(5a-1)/2} = 3a^2$, or, equivalently,

$$\begin{aligned} \binom{\frac{a+3}{2}}{2} s_0 + \binom{\frac{a+1}{2}}{2} s_1 + \binom{\frac{a-1}{2}}{2} s_2 &= 0, \\ \binom{\frac{3a+3}{2}}{2} s_0 + \binom{\frac{3a+1}{2}}{2} s_1 + \binom{\frac{3a-1}{2}}{2} s_2 &= a^2, \\ \binom{\frac{5a+3}{2}}{2} s_0 + \binom{\frac{5a+1}{2}}{2} s_1 + \binom{\frac{5a-1}{2}}{2} s_2 &= 3a^2. \end{aligned}$$

By solving this linear system, s_0, s_1 , and s_2 are given by

$$s_0 = s_2 = \frac{1 - a^2}{8}, \quad s_1 = \frac{a^2 + 3}{3},$$

as they were in (1). Substituting $s_0, s_1,$ and s_2 into (18) leads to

$$\mu_\ell = -\frac{1}{8}(a^2 - (2\ell + 1)^2), \quad \ell \in \mathbb{Z}_+.$$

Finally, by substituting μ_ℓ 's into (19) we arrive at the explicit expressions for ${}_3^a p_k$'s in (5)–(7). This completes the proof of Theorem 1. \square

Proof of Theorem 2.

Similar to the proof of Theorem 1, the two-scale symbol ${}^a P_5$ of ${}^a \phi_5$ must have the form

$${}^a P_5(z) = z^{(1-5a)/2} \left(\frac{1-z^a}{1-z} \right)^5 (s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4)$$

for some constants s_0, \dots, s_4 satisfying $s_4 = s_0, s_3 = s_1,$ and $s_0 + \dots + s_4 = 1.$ First, multiply $s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4$ and $(1-z)^{-5} = \sum_{\ell=0}^{\infty} \binom{4+\ell}{4} z^\ell$ to get

$$\begin{aligned} (s_0 + s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4) (1-z)^{-5} &= \sum_{\ell=0}^{\infty} \nu_\ell z^\ell, \quad \text{where} \\ \nu_\ell &= \binom{\ell+4}{4} s_0 + \binom{\ell+3}{4} s_1 + \binom{\ell+2}{4} s_2 + \binom{\ell+1}{4} s_3 + \binom{\ell}{4} s_4, \quad \ell \in \mathbb{Z}_+. \end{aligned} \quad (20)$$

Secondly, multiply by the expansion of $(1-z^a)^5,$ $\{ {}_5^a p_k \}_{k \in \mathbb{Z}}$ can be expressed in terms of $\{ \nu_\ell \}$ in (20). Then, with $\nu_\ell = 0$ for all $\ell < 0,$ all coefficients of ${}^a P_5(z)$ are now in terms of $s_0, \dots, s_4,$ namely,

$$\begin{aligned} {}_5^a p_k &= \frac{1}{a^4} (\nu_{(5a-1)/2+k} - 5\nu_{(3a-1)/2+k} + 10\nu_{(a-1)/2+k} - 10\nu_{-(a+1)/2+k} \\ &\quad + 5\nu_{-(3a+1)/2+k} - \nu_{-(5+2)/2+k}), \quad |k| \leq (5a-1)/2. \end{aligned} \quad (21)$$

The five requirements

$${}_5^a p_{-2a} = {}_5^a p_{-a} = 0, \quad {}_5^a p_0 = 1, \quad {}_5^a p_a = {}_5^a p_{2a} = 0$$

yield

$$\begin{aligned} \nu_{(a-1)/2} &= 0, \\ \nu_{(3a-1)/2} - 5\nu_{(a-1)/2} &= 0, \\ \nu_{(5a-1)/2} - 5\nu_{(3a-1)/2} + 10\nu_{(a-1)/2} &= a^5, \\ \nu_{(7a-1)/2} - 5\nu_{(5a-1)/2} + 10\nu_{(3a-1)/2} - 10\nu_{(a-1)/2} &= 0, \\ \nu_{(9a-1)/2} - 5\nu_{(7a-1)/2} + 10\nu_{(5a-1)/2} - 10\nu_{(3a-1)/2} + 5\nu_{(a-1)/2} &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \binom{\frac{a+7}{2}}{4} s_0 + \binom{\frac{a+5}{2}}{4} s_1 + \binom{\frac{a+3}{2}}{4} s_2 + \binom{\frac{a+1}{2}}{4} s_3 + \binom{\frac{a-1}{2}}{4} s_4 &= 0, \\ \binom{\frac{3a+7}{2}}{4} s_0 + \binom{\frac{3a+5}{2}}{4} s_1 + \binom{\frac{3a+3}{2}}{4} s_2 + \binom{\frac{3a+1}{2}}{4} s_3 + \binom{\frac{3a-1}{2}}{4} s_4 &= 0, \\ \binom{\frac{5a+7}{2}}{4} s_0 + \binom{\frac{5a+5}{2}}{4} s_1 + \binom{\frac{5a+3}{2}}{4} s_2 + \binom{\frac{5a+1}{2}}{4} s_3 + \binom{\frac{5a-1}{2}}{4} s_4 &= a^4, \\ \binom{\frac{7a+7}{2}}{4} s_0 + \binom{\frac{7a+5}{2}}{4} s_1 + \binom{\frac{7a+3}{2}}{4} s_2 + \binom{\frac{7a+1}{2}}{4} s_3 + \binom{\frac{7a-1}{2}}{4} s_4 &= 5a^4, \\ \binom{\frac{9a+7}{2}}{4} s_0 + \binom{\frac{9a+5}{2}}{4} s_1 + \binom{\frac{9a+3}{2}}{4} s_2 + \binom{\frac{9a+1}{2}}{4} s_3 + \binom{\frac{9a-1}{2}}{4} s_4 &= 15a^4. \end{aligned}$$

Solving this linear system we have s_0, \dots, s_4 in (2), i.e.,

$$\begin{aligned} s_0 = s_4 &= \frac{1}{384}(a^2 - 1)(9a^2 - 1), \\ s_1 = s_3 &= -\frac{1}{96}(a^2 - 1)(9a^2 + 19), \\ s_2 &= \frac{1}{192}(115 + 50a^2 + 27a^4). \end{aligned}$$

Substitute s_0, \dots, s_4 into (20) to get

$$\nu_\ell = \frac{1}{384}(a^2 - (2\ell + 1)^2)(9a^2 - (2\ell + 1)^2), \quad \ell \in \mathbb{Z}_+.$$

Then a_5p_k 's in (8)–(11) subsequently follow. This completes the proof of Theorem 2. □

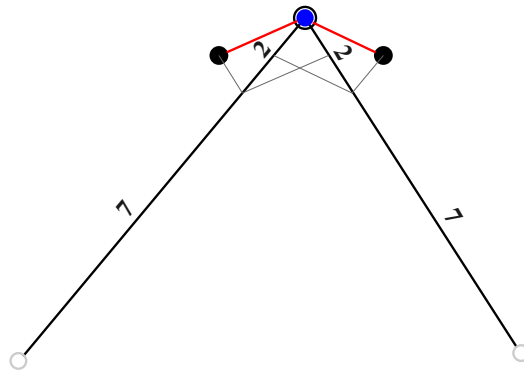


Fig. 3. The geometric illustration of the 3-point ternary subdivision scheme in (22).

4. Applications to Curve Design

With $a = 3$, it follows either from Table I and (5)–(7) or directly from (12)–(14) that the 3-point

ternary interpolatory subdivision scheme is

$$\begin{aligned} \lambda_{3k-1}^{(n+1)} &= \frac{2}{9}\lambda_{k-1}^{(n)} + \frac{8}{9}\lambda_k^{(n)} - \frac{1}{9}\lambda_{k+1}^{(n)}, \\ \lambda_{3k}^{(n+1)} &= \lambda_k^{(n)}, \\ \lambda_{3k+1}^{(n+1)} &= -\frac{1}{9}\lambda_{k-1}^{(n)} + \frac{8}{9}\lambda_k^{(n)} + \frac{2}{9}\lambda_{k+1}^{(n)}, \quad k \in \mathbb{Z}_+. \end{aligned} \tag{22}$$

By observing from (22) that

$$\begin{aligned} \lambda_{3k-1}^{(n+1)} &= \frac{2}{9}\lambda_{k-1}^{(n)} + \frac{7}{9}\lambda_k^{(n)} + \frac{1}{9}(\lambda_k^{(n)} - \lambda_{k+1}^{(n)}), \\ \lambda_{3k+1}^{(n+1)} &= \frac{7}{9}\lambda_k^{(n)} + \frac{2}{9}\lambda_{k+1}^{(n)} + \frac{1}{9}(\lambda_k^{(n)} - \lambda_{k-1}^{(n)}), \quad k \in \mathbb{Z}_+. \end{aligned}$$

the 3-point ternary scheme has a clear geometric interpretation as illustrated by Fig. 3. We also point out that the ternary scheme (22) was also studied in (Hassan & Dodgson [3]) by using the method of “generating function formalism.”

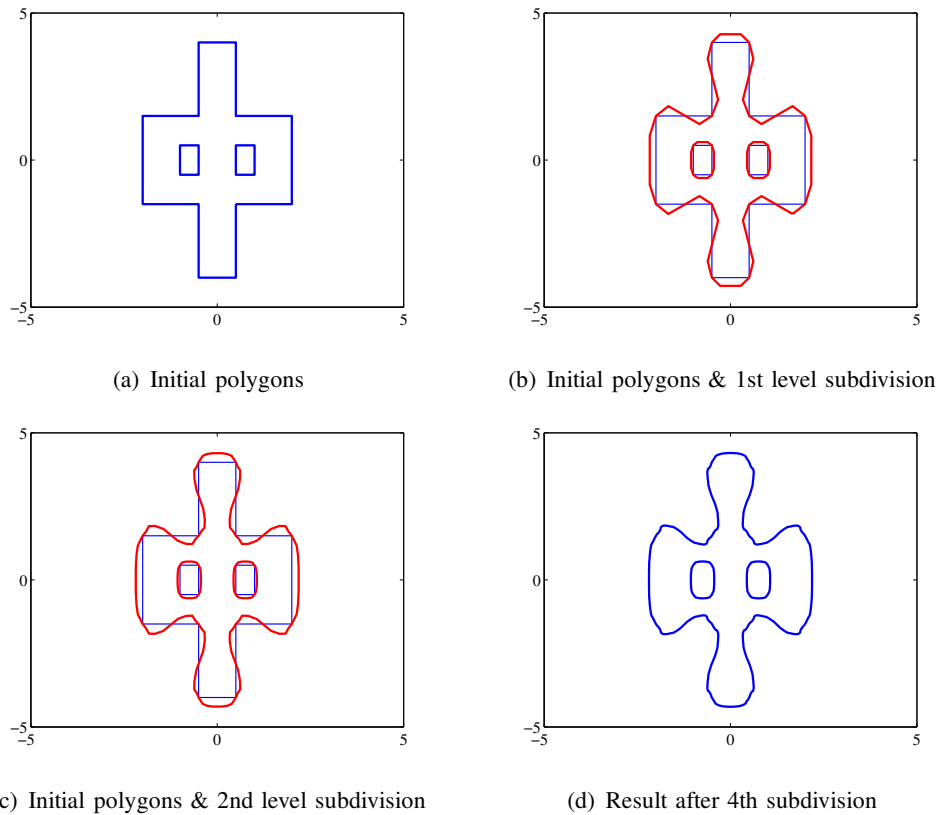


Fig. 4. Three planar polygons with 12, 4, and 4 initial control points.

While when $a = 3$, it follows either from Table II together with (8)–(11) or directly from (15)–

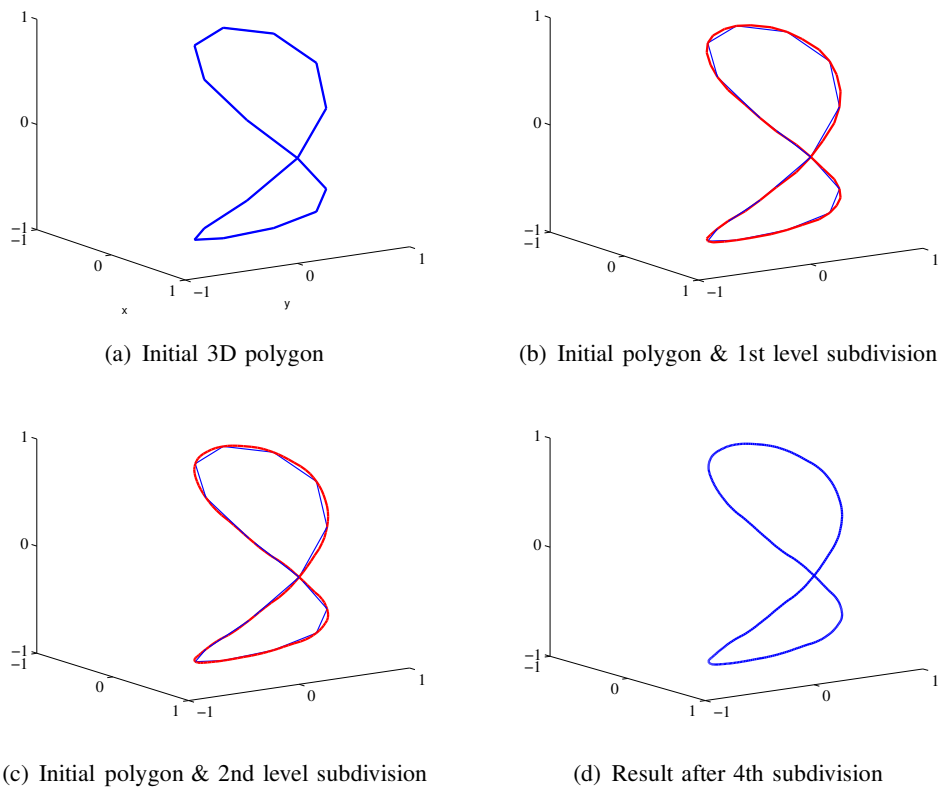


Fig. 5. A space curve with 16 initial control points selected from the Viviani's curve in (24).

(17) that the 5-point ternary interpolatory subdivision scheme is

$$\begin{aligned}
 \lambda_{3k-1}^{(n+1)} &= -\frac{7}{243}\lambda_{k-2}^{(n)} + \frac{70}{243}\lambda_{k-1}^{(n)} + \frac{70}{81}\lambda_k^{(n)} - \frac{35}{243}\lambda_{k+1}^{(n)} + \frac{5}{243}\lambda_{k+2}^{(n)}, \\
 \lambda_{3k}^{(n+1)} &= \lambda_k^{(n)}, \\
 \lambda_{3k+1}^{(n+1)} &= \frac{5}{243}\lambda_{k-2}^{(n)} - \frac{35}{243}\lambda_{k-1}^{(n)} + \frac{70}{81}\lambda_k^{(n)} + \frac{70}{243}\lambda_{k+1}^{(n)} - \frac{7}{243}\lambda_{k+2}^{(n)}, \quad k \in \mathbb{Z}_+.
 \end{aligned}
 \tag{23}$$

To demonstrate the elegance of all these schemes, we apply the 3-point ternary scheme (22) to the 3 closed 2D polygons in Fig. 4(a).

The space polygon in Fig. 5(a) was formed by eight initial control points, selected from the Viviani's curve (Gray [2], p. 201), which is the intersection between a sphere and a right circular cylinder passing through the center of the sphere whose diameter is half of the sphere. Its parametric equation is given by

$$x(t) = \frac{r}{2}(1 + \cos 2t), \quad y(t) = \frac{r}{2} \sin 2t, \quad z(t) = -r \sin t, \quad t \in [0, 2\pi], \tag{24}$$

with r the radius of the sphere. By applying the 3-point ternary scheme (22), the resulting "polygon" after 4th subdivision is shown in Fig. 5(d).

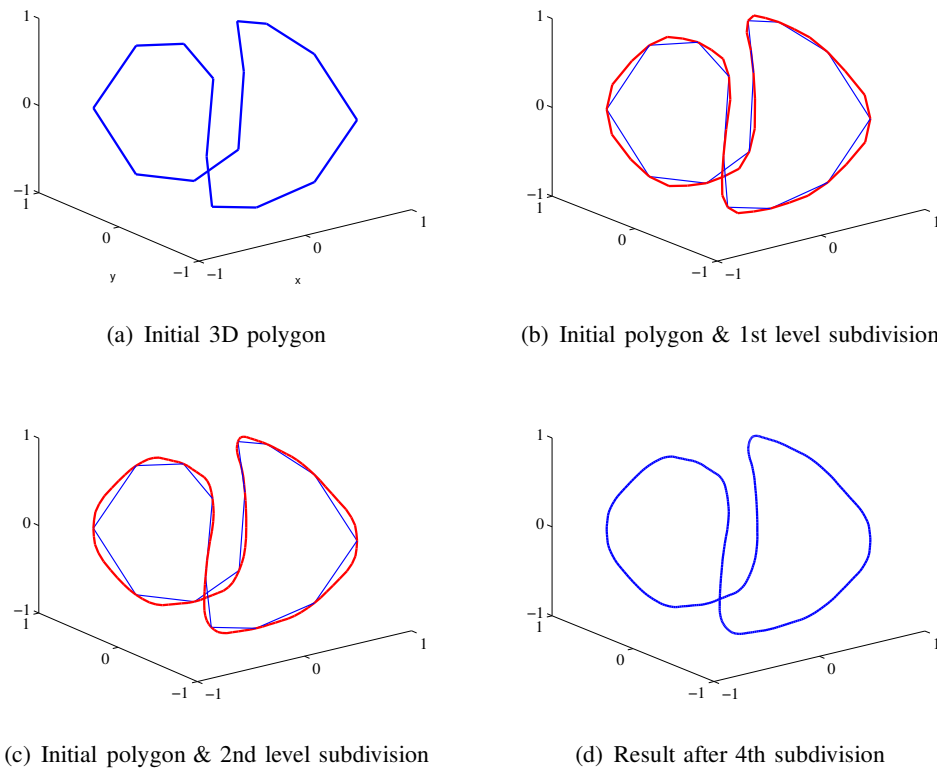


Fig. 6. A closed space curve with 16 initial control points selected from the baseball's seam curve in (25).

A family of curves of a baseball's seam can be given by the following parametric equation

$$\begin{aligned}
 x(t) &= r \cos \left(\left(\frac{\pi}{2} - b \right) \cos 2t \right) \cos (t + b \sin 4t), \\
 y(t) &= r \cos \left(\left(\frac{\pi}{2} - b \right) \cos 2t \right) \sin (t + b \sin 4t), \\
 z(t) &= r \sin \left(\left(\frac{\pi}{2} - b \right) \cos 2t \right), \quad t \in [0, 2\pi],
 \end{aligned} \tag{25}$$

where r is the radius of the baseball, and b is a constant. With the choice of $b = 0.4$, we select 16 points on this curve as shown in Fig. 6(a). We apply the 5-point ternary scheme (23) to get the 3D "curve" in Fig. 6(d) after 4th subdivision.

5. Conclusion

The 3- and 5-point a -ary interpolatory subdivision schemes for curve design were established for any odd integer $a \geq 3$. The polynomial preservation orders of the scaling functions corresponding to these schemes are fixed, namely, either 3 or 5, which is independent of a . The smoothness of the corresponding scaling functions for various values of $a \geq 3$ are needed to and will be studied in detail in the forthcoming paper.

Acknowledgment

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