



## **GLOBAL STABILITY RESULTS OF AN SIS AGE-STRUCTURED EPIDEMIC MODEL WITH VERTICAL TRANSMISSION**

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Received February 12, 2007; revised April 24, 2007; accepted April 25, 2007

### **Abstract**

An SIS age-structured epidemic model for a vertically as well as horizontally transmitted disease is investigated when the fertility, mortality and cure rates depend on age and the force of infection of proportionate mixing assumption type. We determine the steady states and prove the global stability for the endemic equilibriums.

**Keywords:** Vertical transmission; Horizontal transmission; Age-structure; Epidemic; Global stability; Proportionate mixing

**MSC 2000:** 45K05; 45M10; 35A05; 35B30; 35B35; 35B45; 35L40; 92D30; 92D25

### **1. Introduction**

Several recent papers and books have dealt with SIS age-structured epidemic models. In Busenberg, et al. (1988), (1991), (1993a,b) SIS age-structured epidemic models are studied and global stability results are proved. In Iannelli, et al. (1992), some of the previously mentioned works are extended to obtain explicitly computable thresholds and to obtain numerical results. In El-Doma (1999) an SIS age-structured epidemic model is considered and explicitly computable thresholds, stability results as well as the time dependent solutions are given. In El-Doma (2003) the previous stability results are improved and the uniform weak disease persistence is proved. In Busenberg et al. (1996) as well as Langlais et al. (1997) SIS age-structured epidemic models with

dispersal and seasonal periodicities are considered. In Iannelli et al. (1999) an SIS age-structured epidemic model with vital rates which depend on age as well as the population size is considered. In Zhou et al. (2002) an SIS epidemic model with age and infection age is considered. In Inaba (2002), (2001), (1998) an SIS of Pease's evolutionary epidemic model for type A influenza is considered. In Feng et al. (2005) an SIS age-structured epidemic model in a population with multiple groups is considered. For general references see Anderson et al. (1999), Diekmann et al. (2000) and Iannelli (1995).

In this paper, we study an SIS age-structured epidemic model. The disease causes so few fatalities that they can be neglected and is horizontally as well as vertically transmitted. Horizontal transmission of infection is the transfer of infection through some direct or indirect contact with infected individuals, for example, malaria, influenza, gonorrhea and tuberculosis are horizontally transmitted. A particular form of horizontal transmission known as proportionate mixing is assumed in this paper. Vertical transmission of infection is the passing of infection to offspring of infected parentage, for example, AIDS, chagas and hepatitis B are vertically (as well as horizontally) transmitted diseases. This form of transmission plays an important role in maintaining some diseases, for example, see Busenberg et al. (1993b). We note that some infectious diseases such as malaria, influenza, gonorrhea and tuberculosis are of SIS type.

We determine the steady states of the model, under the assumption that the total population has already reached its steady state distribution, and prove global stability

results and show that if  $q \int_0^{\infty} \beta(a) \pi_2(a) da \neq 1$ , see sections 2, 4 for definitions, then the

endemic equilibrium is globally stable. We also show that if  $q \int_0^{\infty} \beta(a) \pi_2(a) da = 1$ , then

either a unique endemic equilibrium exists, and we prove that this endemic equilibrium is globally stable, or the model gives rise to a continuum of endemic equilibria, in the case of non-fertile infectibles, i.e., if individuals are susceptible only after the end of their reproductive period.

We note that the local stability of the endemic equilibrium as well as the global stability of the disease-free equilibrium are reported in EI-Doma (2003), (1999).

The global stability results that we obtain are under very general conditions, and, in fact, we do not require any condition other than the existence of a unique endemic equilibrium. Our results generalize those given in Busenberg, et al. (1988) and also partially improve those given in Iannelli, et al. (1992).

The organization of this paper is as follows: in section 2 we describe the model and obtain the model equations; in section 3 we present preliminary analysis of the model; in section 4 we determine the steady states; in section 5 we prove global stability results; in section 6 we conclude our results.

## 2. The Model

We consider an age-structured population of variable size exposed to a communicable disease. The disease is vertically as well as horizontally transmitted and causes so few fatalities that they can be neglected. We assume the following.

1.  $s(a,t)$  and  $i(a,t)$  respectively, denote the age-density for susceptible and infective of age  $a$  at time  $t$ . Then

$$\int_{a_1}^{a_2} s(a,t) da = \text{total number of susceptible at time } t \text{ of ages between } a_1 \text{ and } a_2,$$

$$\int_{a_1}^{a_2} i(a,t) da = \text{total number of infective at time } t \text{ of ages between } a_1 \text{ and } a_2.$$

We assume that the total population consists entirely of susceptible and infective.

2. Let  $k(a,a')$  denote the probability per unit of time that a susceptible individual of age  $a$  is infected by an infective of age  $a'$ . We further assume that,  $k(a,a') = k_1(a)k_2(a')$  which is known as the "proportionate mixing assumption", see Dietz, et al. (1985). Therefore, the horizontal transmission of the disease occurs at the following rate:

$$k_1(a)s(a,t) \int_0^{\infty} k_2(a')i(a',t) da',$$

where  $k_1(a)$  and  $k_2(a)$  are bounded, nonnegative, continuous functions of  $a$ . The term

$$k_1(a) \int_0^{\infty} k_2(a')i(a',t) da',$$

is called "force of infection" and we let

$$\lambda(t) = \int_0^{\infty} k_2(a)i(a,t) da$$

3. The fertility rate  $\beta(a)$  is a nonnegative, continuous function, with compact support  $[0,A], (A > 0)$ . The number of births of susceptible per unit of time is given by

$$s(0,t) = \int_0^{\infty} \beta(a)[s(a,t) + (1-q)i(a,t)] da, \quad q \in [0,1],$$

where  $q$  is the probability of vertically transmitting the disease. Accordingly all newborns from susceptible parents are susceptible but a portion  $q$  of newborns from infected parents are infective, i.e., they acquire the disease via birth (vertical transmission) and therefore,

$$i(0,t) = q \int_0^{\infty} \beta(a)i(a,t) da.$$

4. The death rate,  $u(a)$ , is the same for susceptible and infective and  $u(a)$  is a non-negative, continuous function and  $\exists a_0 \in [0, \infty)$  such that  $u(a) > \bar{u} > 0, \forall a > a_0$  and  $u(a_2) > u(a_1), \forall a_2 > a_1 > a_0$ .
5. The cure rate  $\gamma(a)$  is a bounded, nonnegative, continuous function of  $a$ . And the cure does not give immunity.
6. The initial age distributions  $s(a, 0) = s_0(a)$  and  $i(a, 0) = i_0(a)$  are continuous nonnegative and integrable functions of  $a \in [0, \infty)$ .

These assumptions lead to the following system of nonlinear integro-partial differential equations with non-local boundary conditions, which describes the dynamics of the transmission of the disease.

$$\left\{ \begin{array}{l} \frac{\partial s(a,t)}{\partial a} + \frac{\partial s(a,t)}{\partial t} + u(a)s(a,t) = -k_1(a)s(a,t)\lambda(t) + \gamma(a)i(a,t), \quad a > 0, t > 0, \\ \frac{\partial i(a,t)}{\partial a} + \frac{\partial i(a,t)}{\partial t} + u(a)i(a,t) = k_1(a)s(a,t)\lambda(t) - \gamma(a)i(a,t), \quad a > 0, t > 0, \\ s(0,t) = \int_0^\infty \beta(a)[s(a,t) + (1-q)i(a,t)] dt, \quad t \geq 0, \\ i(0,t) = q \int_0^\infty \beta(a)i(a,t) da, \quad t \geq 0, \\ \lambda(t) = \int_0^\infty k_2(a)i(a,t) da, \quad t \geq 0, \\ s(a,0) = S_0(a), i(a,0) = i_0(a), \quad a \geq 0. \end{array} \right. \tag{2.1}$$

We note that problem (2.1) is an SIS age-structured epidemic model that has been studied in EI-Doma (2003), (1999), where the steady states are determined and the local asymptotic stability of the endemic equilibrium and the disease-free equilibrium as well as the global stability of the disease-free equilibrium and the time dependent solutions are reported. The same model but with different force of infection term is studied in Busenberg, et al. (1988), (1993), Iannelli, et al. (1992), and the same model, but with  $q = 0$ , the case of no vertical transmission, is studied in Busenberg, et al. (1991).

In what follows, we determine the steady states of the model and prove the global stability of the endemic equilibrium when  $q \int_0^\infty \beta(a)\pi_2(a) da \neq 1$ . We also show that if

$q \int_0^\infty \beta(a)\pi_2(a) da = 1$ , then either a unique endemic equilibrium exists, and we prove that this endemic equilibrium is globally stable, or problem (2.1) gives rise to a continuum of endemic equilibriums in the case of non-fertile infectibles.

### 3. Reduction of the Model

In this section, we develop some preliminary formal analysis of problem (2.1). We define  $p(a,t)$  by  $p(a,t) = s(a,t) + i(a,t)$ . Then from (2.1), by adding the equations, we find that  $p(a,t)$  satisfies the following McKendrick- Von Forester equation:

$$\begin{cases} \frac{\partial p(a,t)}{\partial a} + \frac{\partial p(a,t)}{\partial t} + u(a)p(a,t) = 0, & a > 0, t > 0, \\ p(0,t) = B(t) = \int_0^{\infty} \beta(a)p(a,t) da, & t \geq 0, \\ p(a,0) = p_0(a) = s_0(a) + i_0(a), & a \geq 0. \end{cases} \quad (3.1)$$

Note that problem (3.1) has a unique solution that exists for all time, see Bellman, et al. (1963), Feller (1941) and Hoppensteadt (1975). The unique solution is given by

$$p(a,t) = \begin{cases} p_0(a-t)\pi(a)/\pi(a-t), & a > t \\ B(t-a)\pi(a), & a < t \end{cases} \quad (3.2)$$

where  $\pi(a)$  is given by

$$\pi(a) = e^{-\int_0^a u(\tau) d\tau}$$

and  $B(t)$  has the following asymptotic behavior as  $t \rightarrow \infty$

$$B(t) = [c + \theta(t)]e^{p^*t} \quad (3.3)$$

where  $p^*$  is the unique real number which satisfies the following characteristic equation:

$$\int_0^{\infty} \beta(a)\pi(a)e^{-p^*a} da = 1 \quad (3.4)$$

$\theta(t)$  is a function such that  $\theta(t) \rightarrow 0$  as  $t \rightarrow \infty$  and  $c$  is a constant.

We note that the well posedness of problem (2.1) can be established via the same method as in El- Doma (2005).

#### 4. The Steady States

In this section, we look at the steady state solution of problem (2.1), under the assumption that the total population has already reached its steady state distribution  $p_{\infty}(a) = c\pi(a)$ , i.e., we assume that the characteristic equation (3.4) is satisfied with  $p^* = 0$ , see, for example, Busenberg, et al. (1988).

A steady state  $s^*(a), i^*(a)$  and  $\lambda^*$  must satisfy the following equations:

$$\begin{cases} \frac{ds^*(a)}{da} + u(a)s^*(a) = -k_1(a)s^*(a)\lambda^* + \gamma(a)i^*(a), & a > 0 \\ s^*(0) = c - i^*(0), \end{cases} \tag{4.1}$$

$$\begin{cases} \frac{di^*(a)}{da} + [u(a) + \gamma(a)]i^*(a) = k_1(a)s^*(a)\lambda^*, & a > 0 \\ i^*(0) = q \int_0^\infty \beta(a)i^*(a)da, \end{cases} \tag{4.2}$$

and

$$\lambda^* = \int_0^\infty k_2(a)i^*(a)da \tag{4.3}$$

Anticipating our future needs, we define threshold parameter  $R_0$ , and is given by

$$\begin{aligned} R_0 = c \int_0^\infty \int_0^a k_1(\sigma)k_2(a)\pi(a)e^{-\int_\sigma^a \gamma(\tau)d\tau} d\sigma da \\ + \frac{cq \int_0^\infty \int_0^a \beta(a)\pi(a)k_1(\sigma)e^{-\int_\sigma^a \gamma(\tau)d\tau} d\sigma da \left[ \int_0^\infty k_2(a)\pi_2(a)da \right]}{\left[ 1 - q \int_0^\infty \beta(a)\pi_2(a)da \right]} \end{aligned} \tag{4.4}$$

where  $\pi_2(a)$  is defined by

$$\pi_2(a) = \pi(a)e^{-\int_0^a \gamma(\tau)d\tau} \tag{4.5}$$

Here, we note that the threshold parameter  $R_0$ , usually called the basic reproduction number, and is interpreted as the expected number of secondary cases produced, in a lifetime, by an infective, in a totally susceptible population.

In the following result, we determine the steady state solution of problem (2.1).

**Theorem 4.1:**

(1) If  $R_0 > 1$  and  $q \int_0^\infty \beta(a)\pi_2(a)da \neq 1$  then  $\lambda^* = 0$  and  $\lambda^* > 0$  are possible steady states. A

steady state with  $\lambda^* > 0$  is unique when it exists and satisfies

$$1 = c \int_0^\infty \int_0^a k_1(\sigma)k_2(a)\pi(a)e^{-\int_\sigma^a [\gamma(\tau) + \lambda^*k_1(\tau)]d\tau} d\sigma da$$

$$+ \frac{c q \int_0^a \int_0^a \beta(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\sigma) d\sigma da \left[ \int_0^a k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}{\left[ 1 - q \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}. \quad (4.6)$$

In this case  $s^*(a)$  and  $i^*(a)$  are given by

$$s^*(a) = c\pi(a) - i^*(a), \quad (4.7)$$

$$i^*(a) = i^*(0) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} + c \lambda^* \pi(a) \int_0^a k_1(\sigma) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} d\sigma, \quad (4.8)$$

where  $i^*(0)$  satisfies

$$i^*(0) = \frac{c q \lambda^* \int_0^a \int_0^a \beta(a) \pi(a) k_1(\sigma) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} d\sigma da}{\left[ 1 - q \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]}. \quad (4.9)$$

(2) If  $R_0 \leq 1$ , then the disease-free equilibrium,  $\lambda^* = 0$ , is the only steady state, i.e.,

$$s^*(a) = c\pi(a) \text{ and } i^*(a) = 0$$

(3) If  $q \int_0^a \beta(a) \pi_2(a) da = 1$ , and  $q \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \neq 1$  then  $\exists$  a unique endemic equilibrium.

(4) If  $q \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da = 1$ , then problem (2.1) gives rise to a continuum of endemic equilibria.

**Proof:**

We note that the proofs of (1)-(2) are given in El-Doma (1999). To prove (3), we note that when  $q \int_0^a \beta(a) \pi_2(a) da = 1$ , then  $\int_0^a \beta(a) \pi_2(a) da = 1$ , and accordingly,  $R_0$  is not defined. But if

we assume that  $q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \neq 1$ , then using equations (4.8)-(4.9), and equation (4.3), we obtain the characteristic equation (4.6). We notice that in such case the right-hand side of the characteristic equation (4.6) increases to  $+\infty$  as  $\lambda^* \rightarrow 0$ , and decreases to zero as  $\lambda^* \rightarrow +\infty$ , accordingly,  $\exists$  a unique  $\lambda^* > 0$ , which satisfies the characteristic equation (4.6) and gives rise to a unique endemic equilibrium.

To prove (4), we note that if  $q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da = 1$ , then we first use equation (4.8) and equation (4.2) to obtain the following:

$$i^*(0) = \left[ 1 - q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \right] = cq\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\lambda^* k_1(\tau)]d\tau} k_1(\sigma)d\sigma da \quad (4.10)$$

From equation (4.10) it is easy to see that  $i^*(0)$  is undetermined, and therefore, we use equations (4.3) and (4.8) to obtain the following:

$$\lambda^* \left[ 1 - c \int_0^\infty \int_0^a k_1(\sigma)k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\lambda^* k_1(\tau)]d\tau} d\sigma da \right] = i^*(0) \int_0^\infty k_2(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \quad (4.11)$$

We note that for a fixed  $i^*(0) \in (0, c]$  the left-hand side of equation (4.11) equals zero when  $\lambda^* = 0$ , and increases to  $+\infty$  when  $\lambda^* \rightarrow +\infty$ . Also, the right-hand side of equation (4.11) assumes a positive value when  $\lambda^* = 0$ , and decreases to zero as  $\lambda^* \rightarrow +\infty$ . Accordingly, for each fixed  $i^*(0) \in (0, c]$ , we can see that equation (4.11) gives rise to an endemic equilibrium, and hence problem (2.1) gives rise to a continuum of endemic equilibria in this special case. •

Here, we note that the disease will die out if  $R_0 \leq 1$  and persists if  $R_0 > 1$ . The effect of vertical transmission via its parameter  $q$  is seen, since the right-hand side of equation (4.4) is an increasing function of  $q$  and therefore, a contributing factor for an endemic disease to occur.

### 5. Global Stability Results

In this section, we prove the global stability of the endemic equilibria for problem (2.1). We consider the following transformation, called the age profile of infective:

$$v(a,t) = \frac{i(a,t)}{p_\infty(a)},$$

and note that  $s(a,t) = p_\infty(a) - i(a,t)$ , since we are assuming that the total population has already reached its steady state distribution  $p_\infty(a) = c\pi(a)$ . Therefore, from (2.1),  $v(a,t)$  satisfies the following:

$$\left\{ \begin{array}{l} \frac{\partial v(a,t)}{\partial a} + \frac{\partial v(a,t)}{\partial t} + \gamma(a)v(a,t) = k_1(a)[1-v(a,t)]\lambda(t), \quad a > 0, t > 0, \\ v(0,t) = q \int_0^{\infty} \beta(a)\pi(a)v(a,t)da, \quad t \geq 0, \\ v(a,0) = v_0(a) = i_0(a)/p_{\infty}(a), \quad a \geq 0 \\ \lambda(t) = c \int_0^{\infty} k_2(a)\pi(a)v(a,t)da, \quad t \geq 0 \end{array} \right. \quad (5.1)$$

By integrating (5.1) along characteristic lines  $t-a = \text{const}$ . We find  $v(a,t)$  satisfies

$$v(a,t) \left\{ \begin{array}{l} v_0(a-t)e^{-\int_0^t [\gamma(a-t+\tau)+k_1(a-t+\tau)]\lambda(\tau)d\tau} \\ + \int_0^t k_1(a-t+\sigma)e^{-\int_{\sigma}^t [\gamma(a-t+\tau)+k_1(a-t+\tau)]\lambda(\tau)d\tau} \lambda(\sigma)d(\sigma), \quad a > t, \\ v(0,t-a)e^{-\int_0^a [\gamma(\tau)+k_1(\tau)]\lambda(t-a+\tau)d\tau} \\ + \int_0^a k_1(\sigma)\lambda(t-a+\sigma)e^{-\int_{\sigma}^a [\gamma(\tau)+k_1(\tau)]\lambda(t-a+\tau)d\tau} d\sigma, \quad a < t. \end{array} \right. \quad (5.2)$$

From (5.1),  $v(0,t) = q \int_0^{\infty} \beta(a)\pi(a)v(a,t)da$ , then using equation (5.2) and changing the order of integration several times and making appropriate changes of variables yields

$$v(0,t) = q \left\{ \begin{array}{l} \int_0^t \beta(a)\pi(a)v(0,t-a)e^{-\int_0^a [\gamma(\tau)+k_1(\tau)]\lambda(t-a+\tau)d\tau} da \\ + \int_0^t \int_{\sigma}^{\infty} \beta(a)\pi(a)k_1(a-\sigma)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a [\gamma(\tau)+k_1(\tau)]\lambda(t-a+\tau)d\tau} dad\sigma \\ + \int_t^{\infty} \beta(a)\pi(a)v_0(a-t)e^{-\int_0^t [\gamma(a-t+\tau)+k_1(a-t+\tau)]\lambda(\tau)d\tau} da \end{array} \right\} \quad (5.3)$$

Also, from (5.1)  $\lambda(t) = c \int_0^{\infty} k_2(a)\pi(a)v(a,t)da$ , then using equation (5.2) and changing the order of integration several times and making appropriate changes of variables yields

$$\lambda(t) = c \left\{ \begin{aligned} & \int_0^t k_2(a)\pi(a)v_0(0,t-a)e^{-\int_0^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} da \\ & + \int_0^t \int_\sigma^\infty k_2(a)k_1(a-\sigma)\pi(a)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} dad\sigma \\ & + \int_t^\infty k_2(a)\pi(a)v_0(a-t)e^{-\int_0^t [\gamma(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau} da \end{aligned} \right\} \quad (5.4)$$

Note that by Assumptions 2, 4-6 of section 2 and the Dominated Convergence Theorem, we obtain

$$\int_t^\infty k_2(a)\pi(a)v_0(a-t)e^{-\int_0^t [\gamma(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau} da \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Also, by similar reasoning as above,

$$\int_t^\infty \beta(a)\pi(a)v_0(a-t)e^{-\int_0^t [\gamma(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau} da \rightarrow 0, \text{ as } t \rightarrow \infty.$$

Therefore, setting  $v(0,t) = u(t)$ ,  $u(t)$  and  $\lambda(t)$  satisfy the following limiting equations (see Busenberg, et al. (1988)):

$$u(t) = q \left\{ \begin{aligned} & \int_0^\infty \beta(a)\pi(a)u(t-a)e^{-\int_0^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} da \\ & + \int_0^\infty \int_\sigma^\infty \beta(a)\pi(a)k_1(a-\sigma)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} dad\sigma \end{aligned} \right\}, \quad (5.5)$$

$$\lambda(t) = c \left\{ \begin{aligned} & \int_0^\infty k_2(a)\pi(a)u(t-a)e^{-\int_0^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} da \\ & + \int_0^\infty \int_\sigma^\infty k_2(a)k_1(a-\sigma)\pi(a)\lambda(t-\sigma)e^{-\int_{a-\sigma}^a [\gamma(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau} dad\sigma \end{aligned} \right\} \quad (5.6)$$

We integrate equations (5.5)-(5.6) to obtain the following:

$$\begin{aligned}
u(t) = q \left\{ \int_0^\infty \beta(a) \pi_2(a) u(t-a) e^{-\int_0^a k_1(\tau) \lambda(t-a+\tau) d\tau} da \right. \\
+ \int_0^\infty \beta(a) \pi(a) \left[ 1 - e^{-\int_0^a [\gamma(\tau) + k_1(\tau) \lambda(t-a+\tau)] d\tau} \right] da \\
\left. - \int_0^a \int_\sigma^a \beta(a) \pi(a) e^{-\int_\sigma^a [\gamma(\tau) + k_1(\tau) \lambda(t-a+\tau)] d\tau} \gamma(\sigma) d\sigma da \right\}, \quad (5.7)
\end{aligned}$$

$$\begin{aligned}
\lambda(t) = c \left\{ \int_0^\infty k_2(a) \pi_2(a) u(t-a) e^{-\int_0^a [k_1(\tau) \lambda(t-a+\tau)] d\tau} da \right. \\
+ \int_0^\infty k_2(a) \pi(a) \left[ 1 - e^{-\int_0^a [\gamma(\tau) + k_1(\tau) \lambda(t-a+\tau)] d\tau} \right] da \\
\left. - \int_0^a \int_0^\sigma k_2(a) \pi(a) e^{-\int_0^\sigma [\gamma(\tau) + k_1(\tau) \lambda(t-a+\tau)] d\tau} \gamma(\sigma) d\sigma da \right\}. \quad (5.8)
\end{aligned}$$

Now, we set  $w(t)$  and  $g(t)$  to satisfy the following:

$$\begin{aligned}
w(t) &= u(t) - u^*, \\
g(t) &= \lambda(t) - \lambda^*,
\end{aligned}$$

where  $\lambda^*, u^* = \frac{i^*(0)}{c}$ , are defined as in section 4. Then after some computations, we obtain that  $w(t)$  and  $g(t)$  satisfy the following:

$$\begin{aligned}
w(t) = q \left\{ \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} w(t-a) e^{-\int_0^a k_1(\tau) g(t-a+\tau) d\tau} da \right. \\
+ [1 - u^*] \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} \left[ 1 - e^{-\int_0^a k_1(\tau) g(t-a+\tau) d\tau} \right] da \\
\left. + \int_0^a \int_0^\sigma \beta(a) \pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} \gamma(\sigma) \left[ 1 - e^{-\int_\sigma^a k_1(\tau) g(t-a+\tau) d\tau} \right] d\sigma da \right\}, \quad (5.9)
\end{aligned}$$

$$\begin{aligned}
 g(t) = c & \left\{ \int_0^\infty k_2(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} w(t-a)e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau} da \right. \\
 & + \left[ 1-u^* \right] \int_0^\infty k_2(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} \left[ 1-e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau} \right] da \\
 & \left. + \int_0^\infty \int_0^a k_2(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\lambda^* k_1(\tau)]d\tau} \gamma(\sigma) \left[ 1-e^{-\int_0^a k_1(\tau)g(t-a+\tau)d\tau} \right] d\sigma da \right\}. \tag{5.10}
 \end{aligned}$$

In the following lemma, we give a condition under which  $u^*$  is strictly less than one, and two conditions under which  $u^*$  is equal to one. As we see later, these facts will help us determine some inequalities which in turn will help us determine the global stability of the endemic equilibriums.

**Lemma 5.1.** If  $q \int_0^\infty \beta(a)\pi_2 da \neq 1$ , then  $u^* < 1$ , and if  $q \int_0^\infty \beta(a)\pi_2 da = 1$  and

$$q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \neq 1, \text{ then } u^* = 1.$$

**Proof:** We note that from equation (4.9),  $u^*$  satisfies the following:

$$\begin{aligned}
 1-u^* &= \frac{\left[ 1-q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da - q\lambda^* \int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\lambda^* k_1(\tau)]d\tau} k_1(\sigma) d\sigma da \right]}{\left[ 1-q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \right]} \\
 &= \frac{[1-q] + q \int_0^\infty \int_0^a \beta(a)\pi(a)e^{-\int_0^a [\gamma(\tau)+\lambda^* k_1(\tau)]d\tau} \gamma(\sigma) d\sigma da}{\left[ 1-q \int_0^\infty \beta(a)\pi_2(a)e^{-\lambda^* \int_0^a k_1(\tau)d\tau} da \right]} \tag{5.11}
 \end{aligned}$$

We note that if  $q \int_0^{\infty} \beta(a) \pi_2(a) da \neq 1$ , then from the characteristic equation (3.4) and  $p^* = 0$ , either  $q \neq 1$ , or the support of  $\gamma(a)$  does not lie to the right of the support of  $\beta(a)$ , hence from equation (5.11),  $u^* < 1$ . We also note that if  $q \int_0^{\infty} \beta(a) \pi_2(a) da = 1$ , then from the characteristic equation (3.4) and  $p^* = 0$ , we deduce that  $q = 1$  and the support of  $\gamma(a)$  lies to the right of the support of  $\beta(a)$ , and accordingly, from equation (5.11),  $u^* = 1$ .

In order to facilitate our future calculations, we state the following lemma, the proof of which is by using equation (5.11), lemma 5.1, and straightforward but tedious computation, and therefore, we omit the details of the proof.

**Lemma 5.2:**

Suppose that  $q \int_0^{\infty} \beta(a) \pi_2(a) da \neq 1$ . Then

$$\begin{aligned}
D &= c \left[ 1 - u^* \right] \int_0^{\infty} \int_0^a k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \\
&\quad + c \int_0^{\infty} \int_0^a \int_{\sigma}^a k_2(a) \pi(a) e^{-\int_{\sigma}^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(c) \gamma(\sigma) dc d\sigma da \\
&\quad + \frac{cq \int_0^{\infty} k_2(a) \pi_2(a) da}{\left[ 1 - q \int_0^{\infty} \beta(a) \pi_2(a) da \right]} \left[ 1 - u^* \right] \int_0^{\infty} \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \\
&\quad + \int_0^{\infty} \int_0^a \int_{\sigma}^a \beta(a) \pi(a) e^{-\int_{\sigma}^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(c) \gamma(\sigma) dc d\sigma da \} \\
&= 1 + c\lambda^* \left[ u^* \int_0^{\infty} \int_0^a \int_0^c k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) k_1(c) d\sigma dc da \right. \\
&\quad \left. - \int_0^{\infty} \int_0^a \int_0^c k_2(a) \pi(a) e^{-\int_{\sigma}^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(c) k_1(\sigma) d\sigma dc da \right]
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
 & + \frac{cq\lambda^* \int_0^\infty k_2(a)\pi_2(a) da}{\left[1 - q \int_0^\infty \beta(a)\pi_2(a) da\right]} \left\{ \left[ u^* \right] \int_0^\infty \int_0^a \int_0^c \beta(a)\pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma)k_1(c) d\sigma dc da \right. \\
 & \left. - \int_0^\infty \int_0^a \int_\sigma^a \beta(a)\pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(c)k_1(\sigma) d\sigma dc da \right\} < 1.
 \end{aligned}$$

In the following theorem, we prove the global stability of the endemic equilibrium when  $q \int_0^\infty \beta(a)\pi_2(a) da \neq 1$ . We note that by Lemma 5.1, this condition implies that  $u^* < 1$ .

**Theorem 5.3:** If  $q \int_0^\infty \beta(a)\pi_2(a) da \neq 1$ , then the unique endemic equilibrium, given by Theorem 4.1 (1), is globally stable.

**Proof:**

Let  $|w|^\infty = \limsup_{t \rightarrow \infty} |w(t)|$  and  $|g|^\infty = \limsup_{t \rightarrow \infty} |g(t)|$ . Then if we use the fact that

$$1 - e^{-\int_0^\sigma k_1(\tau)g(t-a+\tau) d\tau} \leq \int_0^\sigma k_1(\tau)g(t-a+\tau) d\tau$$

and then use Fatou's Lemma in equations (5.9), (5.10), we obtain that

$$\begin{aligned}
 |w|^\infty \leq & \frac{q|g|^\infty}{\left[1 - q \int_0^\infty \beta(a)\pi_2(a) da\right]} \left\{ \left[1 - u^*\right] \int_0^\infty \int_0^a \beta(a)\pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \right. \\
 & \left. + \int_0^\infty \int_0^a \int_\sigma^a \beta(a)\pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\tau)\gamma(\sigma) d\tau d\sigma da \right\},
 \end{aligned} \tag{5.13}$$

$$\begin{aligned}
 |g|^\infty \leq & c|w|^\infty \int_0^\infty k_2(a)\pi_2(a) da + \left\{ \left[1 - u^*\right] \int_0^\infty \int_0^a k_2(a)\pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \right. \\
 & \left. + \int_0^\infty \int_0^a \int_\sigma^a k_2(a)\pi(a) e^{-\int_\sigma^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\tau)\gamma(\sigma) d\tau d\sigma da \right\} c|g|^\infty.
 \end{aligned} \tag{5.14}$$

Now, using inequality (5.13) in (5.14), we obtain that

$$|g|^\infty \leq D|g|^\infty. \quad (5.15)$$

And hence, by Lemma 5.2,  $|g|^\infty = 0$ . And then using inequality (5.13), we obtain that  $|w|^\infty = 0$ . Therefore, the endemic equilibrium is globally stable.

In the following lemma, we prove that the constant  $D$ , given in Lemma 5.2, is strictly less than one when the following two conditions are satisfied:

$$q \int_0^\infty \beta(a) \pi_2(a) da = 1, \quad (5.16)$$

$$q \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \neq 1. \quad (5.17)$$

We note that., if condition (5.16) is satisfied then  $R_0$  is not. defined, and in this case either there exists a unique endemic equilibrium when condition (5.17) is also satisfied, which is globally stable, or problem (2.1) gives rise to a continuum of endemic equilibriums, in the case of non-fertile infectibles ( i.e. when the support of  $k_1(a)$  lies to the right of the support of  $\beta(a)$ ).

**Lemma 5.4:**

Suppose that conditions (5.16), (5.17) are satisfied, then the constant  $D$  satisfies the following:

$$\begin{aligned} 0 < D < 1 + c \lambda^* & \left[ \int_0^\infty \int_0^a \int_0^c k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) k_1(c) d\sigma dc da \right. \\ & \left. - \int_0^\infty \int_0^a \int_0^c k_2(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\sigma) k_1(c) d\sigma dc da \right] \\ & + \frac{c \int_0^\infty k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau}}{\left[ 1 - q \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]} \left[ \int_0^\infty \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \right. \\ & \left. - \int_0^\infty \int_0^a \beta(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\sigma) d\sigma da \right] < 1. \end{aligned}$$

**Proof:**

We note that by first assuming that  $q \neq 1$ , and straightforward computations using equations

(5.11), (5.16), (4.6), and condition (5.17), and then letting  $q \rightarrow 1$ , we obtain

$$\begin{aligned}
 0 < D = & 1 - c \int_0^\infty \int_0^a k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \\
 & - c \lambda^* \int_0^\infty \int_0^a \int_0^c k_2(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\sigma) k_1(c) d\sigma dc da \\
 & + \frac{c \int_0^\infty k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da}{\left[ 1 - \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]} \left[ \int_0^\infty \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \right. \\
 & \left. - \int_0^\infty \int_0^a \beta(a) \pi(a) e^{-\int_0^a [\gamma(\tau) + \lambda^* k_1(\tau)] d\tau} k_1(\sigma) d\sigma da \right] \\
 & + \frac{\lambda^* \int_0^\infty \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da}{\left[ 1 - \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]} \times \left( c \int_0^\infty \int_0^a k_2(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da \right).
 \end{aligned}
 \tag{5.18}$$

Now, we notice that by equation (5.11) and Lemma 5.1,

$$\frac{\lambda^* \int_0^\infty \int_0^a \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} k_1(\sigma) d\sigma da}{\left[ 1 - \int_0^\infty \beta(a) \pi_2(a) e^{-\lambda^* \int_0^a k_1(\tau) d\tau} da \right]} < 1,$$

and accordingly the conclusion of the Lemma follows from (5.18).

In the following theorem, we will prove that the endemic equilibrium, given by Theorem 4.1 (3), is globally stable.

**Theorem 5.5:** Suppose that conditions (5.16), (5.17) are satisfied. Then the endemic equilibrium, given by Theorem 4.1 (8), is globally stable.

**Proof:**

We note that from Lemma 5.4, we obtain that  $|g|^\infty \leq D|g|^\infty < |g|^\infty$ , and hence,  $|g|^\infty = 0$ . Now, using  $|g|^\infty = 0$  in inequality (5.13), we obtain that  $|w|^\infty = 0$ . Accordingly, we obtain the global stability for the endemic equilibrium.

## 6. Conclusion

We studied an SIS age-structured epidemic model when the disease is vertically as well as horizontally transmitted and the force of infection of proportionate mixing assumption type. The mortality, fertility and cure rates are age-dependent. We note that malaria, influenza, gonorrhea and tuberculosis are examples of SIS epidemics. We determined the steady states of the model and proved global stability results for the endemic equilibria. If

$q \int_0^\infty \beta(a)\pi_2(a)da \neq 1$ , then the endemic equilibrium is globally stable. And if

$q \int_0^\infty \beta(a)\pi_2(a)da = 1$ , then either a unique endemic equilibrium exists, and we proved that this

endemic equilibrium is globally stable, or the model gives rise to a continuum of endemic equilibria, if individuals are susceptible only after the end of their reproductive period.

The global stability results that we obtained are under very general conditions, and, in fact, we did not require any condition other than the existence of a unique endemic equilibrium. Our results generalized those given in Busenberg, et al. (1988) and also partially improved those given in Iannelli, et al. (1992).

## Acknowledgments

*This work was completed while the author was an Arab Regional Fellow at the Center for Advanced Mathematical Sciences (CAMS), American University of Beirut, Beirut, Lebanon, he was supported by a grant from the Arab Fund for Economic and Social Development, and he would like to thank the Director of CAMS, Prof. Dr. Wafic Sabra, for an invitation and hospitality during his stay in CAMS. Also, the author would like to thank Prof. Dr. Mimmo Iannelli and Prof. Dr. Odo Diekmann for their advice and for sending references, and Dr. Hisashi Inaba for providing references.*

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