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Interval – mtype Oscillation Criteria for Half – Linear PDE with Damping

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Abstract

Using the Riccati substitution we derive new sufficient conditions which ensure that the half-linear partial differential equation with p-Laplacian and damping in the form

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+\left\langle \vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle + c(x)|u|^{p-2}u=0, \tag{E}$$

is oscillatory. These criteria, called interval criteria in theory of ODE's, allow to eliminate "bad parts" of the potential function $c(x)$ from our considerations. Some of the results are new even in the case when (E) becomes linear ordinary differential equation.

Key words: p-Laplacian, oscillatory solution, Riccati equation, interval criteria, averaging technique

AMS Class: 35B05, 35J15, 35J60

1. Introduction

In the paper we investigate the equation with p-Laplacian and Emden-Fowler type nonlinearity

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u)+\left\langle \vec{b}(x),\|\nabla u\|^{p-2}\nabla u\right\rangle + c(x)|u|^{p-2}u=0, \tag{E}$$

where p is a real number satisfying $p>1$, $\|\cdot\|$ is the usual Euclidean norm and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^n$. The potential function $c(x)$ and damping function $\vec{b}(x)$ are supposed to be locally Hölder continuous. Among others, we do not assume anything concerning either the fixed sign or the radial symmetry of the potential $c(x)$.

Equation (E) is sometimes referred as a half-linear equation, since a constant multiple of every solution is a solution of the same equation. From this reason many of the qualitative properties of half-linear equation (E) are similar to the properties of linear Schrödinger partial differential equation

$$\Delta u + c(x)u = 0 \tag{1}$$

which can be obtained from (E) for $p=2$. Especially the Sturmian type theorems extend from (1) also to (E), see Jaroš, et al. (2000) and Došlý and Mařík (2001).

Notation: In the sequel we denote by $S(a)$, $\Omega(a)$, $\Omega(a,b)$ and D the sets in $\mathbb{R}^n$ and $\mathbb{R}^2$ as follows:

$$\begin{aligned} S(a) &= \{x \in \mathbb{R}^n : a = \|x\|\}, \\ \Omega(a) &= \{x \in \mathbb{R}^n : a \le \|x\|\}, \\ \Omega(a,b) &= \{x \in \mathbb{R}^n : a \le \|x\| \le b\}, \\ D &= \{(t,s) \in \mathbb{R}^2 : t \ge s\}. \end{aligned}$$

Throughout the paper, q denotes the conjugate number to p, i.e., $q = \frac{p}{p-1}$ and $\vec{\nu}(x)$ denotes the outside normal unit vector to the sphere $S(x)$, i.e. $\vec{\nu}(x) = (x_i)\|x\|^{-1}$. Finally, $\mathrm{d}\sigma$ is the integral element of the sphere $S(x)$.

The oscillation theory of (1) deals with two types of oscillation. According to this theory, equation (1) is said to be *weakly oscillatory* if every its solution has a zero outside of every ball in $\mathbb{R}^n$ and *strongly oscillatory* if every solution has a nodal domain outside of every ball in $\mathbb{R}^n$. Moss and Piepenbrick (1978) showed that both definitions are equivalent if the function $c(x)$ is locally Hölder continuous. As far as the author knows, the possible equivalence between both types of oscillation remains an open question for equation (E). In this paper the first type of oscillation is used, as the following definition shows.

Definition 1: Let Ω be unbounded domain in $\mathbb{R}^n$. Equation (E) is said to be oscillatory in Ω if every its nontrivial solution defined on $\Omega \cap \Omega(t_0)$ has zero in $\Omega \cap \Omega(t)$ for every $t \ge t_0$. Equation (E) is said to be oscillatory, if it is oscillatory in $\mathbb{R}^n$.

Kong (1999) used the Riccati technique and the two-parametric averaging function $H(t,s)$ (a technique originally due to Philos (1989)) to obtain new conjugacy criteria for linear second order ordinary differential equation

$$(p(t)y')' + q(t)y = 0 \qquad (2)$$

and derived sufficient conditions which guarantee existence of infinitely many intervals with pairs of conjugate points. These conditions allow to eliminate "bad parts" of the interval (t_0,∞) from the oscillation criteria and are applicable even if the integral of the function $q(t)$ is small, e.g. if $\int_0^\infty q(t)\mathrm{d}t = -\infty$. The results of Kong (1999) have been extended by Wang and Yang (2004) for half-linear ODE.

The aim of this paper is to extend the criteria from Kong (1999) and Wang and Yang (2004) for equation (E). In addition, we offer an improvement of these results (see Remark 3 below) which is new even in the case of the original equation (2) and it is closely related to the recent result of Sun (2004).

Oscillation properties of equation (E) and several (less or more general) similar equations have been studied by Riccati technique (see Lemma 1 below) in a series of papers by Xu (2005) and Xu, Xing (2003, 2004). In these papers authors, starting with integration of the Riccati equation over spheres in $\mathbb{R}^n$ centered in the origin, convert the n-dimensional problem into a problem in one variable and then employ the corresponding techniques from the oscillation theory of ordinary differential equations. The oscillation criteria obtained in this way detect the oscillation only if the mean value of the potential function $c(x)$ over the spheres centered in the origin is “sufficiently large”.

In this paper, like already in Mařík (2004), we prefer an advanced approach than that one used in papers by Xu (2005) we use the averaging function which need not to preserve radial symmetry. As a particular example, we use the (n+1)-variable function $H(t,\|x\|)\rho(x)$, where $x\in\mathbb{R}^n$, rather than the function of two variables $H(t,s)k(s)$ with $s\in\mathbb{R}$, used in Xu and Xing (2003), where s corresponds to our $\|x\|$. Following this approach we obtain oscillation criteria which are applicable also to the cases when the equation is strongly asymmetric with respect to origin and the mean value of the potential $c(x)$ is small, as has been explained in Mařík (2004, Remark 2.3). The author believes that such a criteria are more natural for partial differential equations and provide much deeper insight into oscillation. Moreover, the oscillation of radially symmetric PDE's can be studied in the scope of ODE's and oscillation of PDE's with ``sufficiently large'' mean value of the potential function can be detected via oscillation of certain ordinary differential equation, as has been proved independently by Jaroš et al. (2000) and by Došlý and Mařík (2001).

The method presented in this paper is applicable also to several similar equations, like (if $p=2$) the nonlinear equation

$$\Delta u+\left\langle \vec{b}(x),\nabla u\right\rangle + c(x)f(u)=0, \qquad (3)$$

where the continuous function f satisfies sign condition $uf(u)>0$ for $u\neq 0$ and equation (3) is a “Sturmian majorant” of (E), e.g. if $f'(u)>\mu>0$ for some $\mu\in\mathbb{R}$ and every $u>0$. However, in order to make our ideas more transparent, we keep the term $c(x)|u|^{p-2}u$ rather than replacing this term by a term of the type $c(x)f(u)$ and hence consider the simpler equation (E) only.

The following lemma introduces our main tool – Riccati-type substitution which converts equation (E) into first order Riccati-type equation.

Lemma 1: 1 Let u be a solution of equation (E) which has no zero on the domain $\Omega\subseteq\mathbb{R}^n$. Then the vector variable $\vec{w}(x)$ defined on the domain Ω by

$$\vec{w}(x)=\frac{\|\nabla u(x)\|^{p-2}\nabla u(x)}{|u(x)|^{p-2}u(x)} \qquad (4)$$

solves the Riccati-type equation

$$\operatorname{div}\vec{w}+c(x)+(p-1)\|\vec{w}\|^{q}+\left\langle \vec{w},\vec{b}(x)\right\rangle=0 \qquad (5)$$

on Ω.

Proof: By direct substitution, see Mařík (2004).

In the sequel we define two classes of averaging functions: each of them will be used on one of the parts of boundary $\partial\Omega(a,b)=S(a)\cup S(b)$.

Definition 2: The function $H(t,s)\in C(D,[0,\infty))$ is said to belong to the class $\mathcal{H}$ if

(i) $H(t,s)=0$ if and only if $t=s$.

(ii) The partial derivative $\frac{\partial H}{\partial s}(t,s)$ exists.

(iii) 1 Denoting

$$h_2(t,s) = -\frac{\partial H}{\partial s}(t,s)H^{-1}(t,s), \quad \text{for } (t,s)\in D,\ t\neq s,$$

the function $h_2^p(t,s)H(t,s)$ is locally integrable on each compact subset in D.

Remark 1: Remember that the function $h_2(t,s)$ has singularity for $s=t$, since $H(t,t)=0$. The same is true also for the function $h_1^*(t,s)$ defined below.

Definition 3: The function $H^*(t,s)\in C(D,[0,\infty))$ is said to belong to the class $\mathcal{H}^*$ if

(i) $H^*(t,s)=0$ if and only if $t=s$.

(ii) The partial derivative $\frac{\partial H^*}{\partial t}(t,s)$ exists

(iii) Denoting

$$h_1^*(t,s) = \frac{\partial H^*}{\partial t}(t,s)[H^*(t,s)]^{-1}, \qquad \text{for } (t,s)\in D,\ t\neq s,$$

the function $[h_1^*(t,s)]^p H^*(t,s)$ is locally integrable on each compact subset in D.

Remark 2: Note that the functions h_1^*, h_2 play slightly different role in this paper than in the paper Wang, Yang (2004) where $h_2(t,s) = \frac{\partial H}{\partial s}(t,s)H^{-1/2}(t,s)$ and h_1^* is defined in the similar way. The reason is that we wish to gain simpler formulas in our resulting oscillation criteria.

2. Auxiliary results

Lemma 2: Let u be a solution of (E) such that $u(x)>0$ for $c\leq\|x\|<b$. Let $\rho\in C^1(\mathbb{R}^n,\mathbb{R}^+)$ be a smooth positive function and H be a function of the class $\mathcal{H}$. The vector variable $\vec{w}(x)$ defined by (4) satisfies the inequality

$$\begin{aligned}\int_{\Omega(c,b)} H(b,\|x\|)c(x)\rho(x)\mathrm{d}x &\leq H(b,c)\int_{S(c)}\rho(x)\langle\vec{w}(x),\vec{\nu}(x)\rangle\mathrm{d}\sigma \\ &+\int_{\Omega(c,b)}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2(b,\|x\|)\vec{\nu}\right\|^p \rho(x)H(b,\|x\|)p^{-p}\,\mathrm{d}x.\end{aligned} \tag{6}$$

Proof: Suppose that positive solution u of (E) exists for $c\leq\|x\|<b$. Multiplying the Riccati equation (5) by $\rho(x)$ we get

$$c(x)\rho(x) = -\rho(x)\operatorname{div}\vec{w}-(p-1)\rho(x)\|\vec{w}\|^q-\langle\rho(x)\vec{w},\vec{b}(x)\rangle$$

and hence

$$c(x)\rho(x) = -\operatorname{div}(\rho(x)\vec{w})-(p-1)\rho(x)\|\vec{w}\|^q-\left\langle\rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle. \tag{7}$$

Integrating over the sphere $S(s)$ of radius s, multiplying by $H(t,s)$ and integrating with respect to s over the interval (c,t), where $t<b$, we get

$$\begin{aligned}\int_{\Omega(c,t)} H(t,\|x\|)\rho(x)c(x)\mathrm{d}x &= -\int_c^t H(t,s)\int_{S(s)} \mathrm{div}(\rho(x)\vec{w})\mathrm{d}\sigma\,\mathrm{d}s\\ &-(p-1)\int_{\Omega(c,t)} H(t,\|x\|)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x\\ &-\int_{\Omega(c,t)} H(t,\|x\|)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x.\end{aligned}$$

Integration by parts in the first integral on the right hand side, Gauss-Ostrogradskii formula and the definition of the function h_2 give

$$\begin{aligned}\int_{\Omega(c,t)} H(t,\|x\|)c(x)\rho(x)\mathrm{d}x &= H(t,c)\int_{S(c)} \rho(x)\langle\vec{w},\vec{\nu}\rangle\mathrm{d}\sigma\\ &-\int_{\Omega(c,t)} h_2(t,\|x\|)H(t,\|x\|)\rho(x)\langle\vec{w},\vec{\nu}\rangle\mathrm{d}x\\ &-(p-1)\int_{\Omega(c,t)} H(t,\|x\|)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x\\ &-\int_{\Omega(c,t)} H(t,\|x\|)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x.\end{aligned}$$

Using the well-known Young inequality

$$\frac{\|\vec{X}\|^q}{q}\pm\langle\vec{X},\vec{Y}\rangle+\frac{\|\vec{Y}\|^p}{p}\geq 0 \tag{8}$$

with

$$\vec{X} = \rho(x)((p-1)H(t,\|x\|))^{\frac{1}{q}}\rho^{\frac{1}{q}-1}(x)q^{\frac{1}{q}}\vec{w}$$

and

$$\vec{Y} = H(t,\|x\|)(\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2(t,\|x\|)\vec{\nu})((p-1)H(t,\|x\|))^{-\frac{1}{q}}\rho^{1-\frac{1}{q}}(x)q^{-\frac{1}{q}}$$

and using the obvious identities $(p-1)q=p$, $p\left(1-\frac{1}{q}\right)=1$ we get

$$\begin{aligned}\int_{\Omega(c,t)} H(t,\|x\|)c(x)\rho(x)\mathrm{d}x\leq\ & H(t,c)\int_{S(c)} \rho(x)\langle\vec{w},\vec{\nu}\rangle\mathrm{d}\sigma\\ &+\int_{\Omega(c,t)} \frac{1}{p}H^p(t,\|x\|)\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2(t,\|x\|)\vec{\nu}\right\|^p\\ &\times\rho(x)(pH(t,\|x\|))^{-p/q}\,\mathrm{d}x.\end{aligned}$$

Now some easy simplifications, identity $\frac{p}{q}=p-1$ and limit process $t\to b^-$ give (6).

Lemma 3: Let u be a solution of (E) such that $u(x)>0$ for $a<\|x\|\leq c$. Let $\rho\in C^1(\mathbb{R}^n,\mathbb{R}^+)$ be a smooth positive function and H^* be a function of the class $\mathcal{H}^*$. The vector variable $\vec{w}(x)$ defined by (4) satisfies the inequality

$$\int_{\Omega(a,c)} H^*(\|x\|,a)\rho(x)c(x)\mathrm{d}x \le -H^*(c,a)\int_{S(c)}\rho(x)\langle \vec{w}(x),\vec{\nu}\rangle \mathrm{d}\sigma$$
$$+\int_{\Omega(a,c)}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*(\|x\|,a)\vec{\nu}\right\|^p \rho(x)H^*(\|x\|,a)\,p^{-p}\,\mathrm{d}x. \tag{9}$$

Proof: We begin as in the proof of Lemma 2 and obtain (7). Integrating (7) over the sphere $S(s)$ of radius s, multiplying by $H^*(s,t)$ and integrating with respect to s over the interval (t,c), where $a<t$, we get

$$\int_{\Omega(t,c)} H^*(\|x\|,t)c(x)\rho(x)\mathrm{d}x = -\int_t^c H^*(s,t)\int_{S(s)}\mathrm{div}(\rho(x)\vec{w})\mathrm{d}\sigma\,\mathrm{d}s$$
$$-(p-1)\int_{\Omega(t,c)} H^*(\|x\|,t)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x$$
$$-\int_{\Omega(t,c)} H^*(\|x\|,t)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x.$$

As in the proof of Lemma 3, the integration by parts in the first integral on the right hand side, Gauss-Ostrogradskii formula and the definition of the function h_1^* give

$$\int_{\Omega(t,c)} H^*(\|x\|,t)\quad c(x)\rho(x)\mathrm{d}x = -H^*(c,t)\int_{S(c)}\rho(x)\langle \vec{w},\vec{\nu}\rangle \mathrm{d}\sigma$$
$$+\int_{\Omega(t,c)} h_1^*(\|x\|,t)H^*(\|x\|,t)\rho(x)\langle \vec{w},\vec{\nu}\rangle \mathrm{d}x$$
$$-(p-1)\int_{\Omega(t,c)} H^*(\|x\|,t)\rho(x)\|\vec{w}\|^q\,\mathrm{d}x$$
$$-\int_{\Omega(t,c)} H^*(\|x\|,t)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle \mathrm{d}x.$$

Young inequality (8) with

$$\vec{X} = \rho(x)((p-1)H^*(\|x\|,t))^{\frac{1}{q}}\rho^{\frac{1}{q}-1}(x)q^{\frac{1}{q}}\vec{w}$$

and

$$\vec{Y} = H^*(\|x\|,t)(\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*(\|x\|,t)\vec{\nu})((p-1)H^*(\|x\|,t))^{-\frac{1}{q}}\rho^{1-\frac{1}{q}}(x)q^{-\frac{1}{q}},$$

some simplifications and limit process $t\to a^+$ give (9), similarly as in the proof of Lemma 3.

3. Main Results

Theorem 1: Suppose that there exist real number $c\in(a,b)$, positive smooth function $\rho(x)$ and averaging functions $H(t,s)\in\mathcal{H}$, $H^*(t,s)\in\mathcal{H}^*$, such that

$$\frac{1}{H^*(c,a)}\int_{\Omega(a,c)} H^*(\|x\|,a)\rho(x)c(x)\mathrm{d}x+\frac{1}{H(b,c)}\int_{\Omega(c,b)} H(b,\|x\|)\rho(x)c(x)\mathrm{d}x$$
$$>\frac{1}{H^*(c,a)}\int_{\Omega(a,c)}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*(\|x\|,a)\vec{\nu}\right\|^p \rho(x)H^*(\|x\|,a)\,p^{-p}\,\mathrm{d}x \tag{10}$$

$$+\frac{1}{H(b,c)}\int_{\Omega(c,b)}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2(b,\|x\|)\vec{\nu}\right\|^p\rho(x)H(b,\|x\|)p^{-p}\,\mathrm{d}x.$$

Then every solution of (E) has at least one zero inside $\Omega(a,b)$.

Proof: Suppose, by contradiction, that a solution u with no zero in the interior of $\Omega(a,b)$ exists. Without loss of generality we can suppose that the function c is positive inside $\Omega(a,b)$. Then (6) and (9) hold. Dividing these inequalities by $\frac{1}{H(b,c)}$ and $\frac{1}{H^*(c,a)}$ respectively and summing up we obtain an opposite inequality to (10). This contradiction shows that Theorem 1 holds.

Theorem 2: If there exist $t_0>0$, $H\in\mathcal{H}$, $H^*\in\mathcal{H}^{\square}$, $\rho\in C^1(\Omega(t_0),\mathbb{R}^+)$ such that for every $\tau>t_0$ the inequalities

$$\limsup_{t\to\infty}\int_{\Omega(\tau,t)}[H(t,\|x\|)\rho(x)c(x)-\frac{\rho(x)H(t,\|x\|)}{p^p}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+h_2(t,\|x\|)\vec{\nu}\right\|^p]\mathrm{d}x>0 \qquad (11)$$

and

$$\limsup_{t\to\infty}\int_{\Omega(\tau,t)}[H^*(\|x\|,\tau)\rho(x)c(x)-\frac{\rho(x)H^*(\|x\|,\tau)}{p^p}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}-h_1^*(\|x\|,\tau)\vec{\nu}\right\|^p]\mathrm{d}x>0. \qquad (12)$$

hold, then equation (E) is oscillatory.

Main idea of the proof: If the assumptions of Theorem 2 hold, then for every $T>t_0$ there exist numbers $a<c<b$ such that (10) holds and hence the equation has arbitrarily large zeros. Here we omit the details, since the proof is completely analogous to the one-dimensional case, see e.g. Wang, Yang (2004, Theorem 3).

Remark 3: If $n=1$, $\vec{b}=\vec{o}$ and $H(t,s)=H^*(t,s)$, then Theorem 1 correspond to Wang, Yang (2004, Theorem 3) with $r\equiv 1$. Remark that, as far as the author knows, all relevant results in the literature suppose $H(t,s)=H^*(t,s)$, i.e. the same weighting function is used on both ends of the interval (a,b). Hence the possibility $H(t,s)\not\equiv H^*(t,s)$ causes that Theorem 2 is new even for linear ODE (2). Another, very similar, approach which allows to use different weighting functions on both ends of the interval (a,b) has been presented by Sun (2004) for $n=1$ and by Xu (2005) for $n\geq 2$. Namely, these authors use the function $\widehat{H}(r,s,l)$ of three variables which corresponds, in some sense, to our product $H(r,s)H^*(s,l)$.

In the following theorem we utilize this idea and use the product $H(t_2,\|x\|)H^*(\|x\|,t_1)$ as an averaging function in the procedure from Lemma 2. As a result we obtain an oscillation criterion which is simpler than 11 – (12) in the sense that it consists of one inequality only, but it contains more complicated function in the integral. This theorem is an n-dimensional extension of Sun (2004, Theorem 2.5) and non-radial extension of Xu (2005, Theorem 2.2) with slightly different meaning of h_1^*, h_2, as mentioned above.

Theorem 3: Suppose that for every $T > t_0$ there exist $t_1 > T$, $H \in \mathcal{H}$ and $H^* \in \mathcal{H}^*$ such that

$$\limsup_{t\to\infty}\int_{\Omega(t_1,t)} H(t,\|x\|)H^*(\|x\|,t_1)[\rho(x)c(x) - \frac{\rho(x)}{p^p}\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+[h_2(t,\|x\|)-h_1^*(\|x\|,t_1)]\vec{\nu}\right\|^p]\mathrm{d}x > 0. \quad (13)$$

Proof: As in the proof of Lemma 2 we get (7). Integrating over the sphere $S(s)$ of radius s, multiplying by $H(t,s)H^*(s,t_1)$ and integrating with respect to s over the interval (t_1,t) we get

$$\begin{aligned}\int_{\Omega(t_1,t)} & H(t,\|x\|)H^*(\|x\|,t_1)\rho(x)c(x)\mathrm{d}x \\ &= -\int_{t_1}^{t} H(t,s)H^*(s,t_1)\int_{S(s)}\mathrm{div}(\rho(x)\vec{w})\mathrm{d}\sigma\,\mathrm{d}s \\ &\quad -(p-1)\int_{\Omega(t_1,t)} H(t,\|x\|)H^*(\|x\|,t_1)\rho(x)\|\vec{w}\|^q\mathrm{d}x \\ &\quad -\int_{\Omega(t_1,t)} H(t,\|x\|)H^*(\|x\|,t_1)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle\mathrm{d}x.\end{aligned}$$

Integration by parts in the first integral on the right hand side, Gauss-Ostrogradskii formula and the definition of the function h_2 give

$$\begin{aligned}\int_{\Omega(t_1,t)} & H(t,\|x\|) \quad H^*(\|x\|,t_1)\rho(x)c(x)\mathrm{d}x \\ = &\quad -\int_{\Omega(t_1,t)}[h_2(t,\|x\|)-h_1^*(\|x\|,t_1)] \\ &\quad \times H(t,\|x\|)H^*(\|x\|,t_1)\rho(x)\langle\vec{w},\vec{\nu}\rangle\mathrm{d}x \\ &\quad -(p-1)\int_{\Omega(t_1,t)} H(t,\|x\|)H^*(\|x\|,t_1)\rho(x)\|\vec{w}\|^q\mathrm{d}x \\ &\quad -\int_{\Omega(t_1,t)} H(t,\|x\|)H^*(\|x\|,t_1)\left\langle \rho(x)\vec{w},\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}\right\rangle\mathrm{d}x.\end{aligned}$$

The Young inequality yields

$$\begin{aligned}\int_{\Omega(t_1,t)} & H(t,\|x\|)H^*(\|x\|,t_1)\rho(x)c(x)\mathrm{d}x \le \int_{\Omega(t_1,t)}\frac{1}{p}[H(t,\|x\|)H^*(\|x\|,t_1)]^p \\ &\times\left\|\vec{b}(x)-\frac{\nabla\rho(x)}{\rho(x)}+[h_2(t,\|x\|)-h_1^*(\|x\|,t_1)]\vec{\nu}\right\|^p \\ &\times\rho(x)(pH(t,\|x\|)H^*(\|x\|,t_1))^{-p/q}\mathrm{d}x.\end{aligned}$$

Using some algebraic simplifications we find that the integral from the left hand side of (13) is bounded from above by zero for every $t > t_1$ which contradicts the assumption (13). Theorem is proved.

Remark 4: The sharpness of the method presented in this paper can be shown on convenient examples of radially symmetric equations which follow the corresponding examples for $n = 1$ and therefore we omitted the details.

Several effective criteria can be derived from criteria in this paper by choosing particular averaging functions. The most typical functions of the classes $\mathcal{H}$ and $\mathcal{H}^*$ are

$$H(t,s) = (t-s)^{\alpha}, \qquad \text{and} \qquad H^*(t,s) = (t-s)^{\beta},$$

where $\min\{\alpha, \beta\} > p-1$ (this restriction follows from the condition (1)). With this averaging functions the oscillation criteria are called Kamenev-type criteria.

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